Exercises on chapter 4

Exercise 1: OLG model with a CES production function This exercise studies the dynamics of the standard OLG model with a utility function given by:

$$U(c_t, d_{t+1}) = (1-a)\ln c_t + a\ln d_{t+1}$$

and a CES production function:

$$F(K,L) = A\left(K^{-1} + L^{-1}\right)^{-1}$$

All other assumptions are as in the course. Find the dynamics of the variable $k_t = K_t/N_t$.

Prove that if $aA > 4(1+n)$, there exists three steady states (0 and two positive steady states). Show that there exists a poverty trap for k_0 to small.

Exercise 2: OLG model with agents living 3 periods

A model with overlapping generations is considered, in which N_t agents are born at each period t. N_t increases at a constant rate n : $N_t = (1+n)N_{t-1}$. Each agent is living during 3 periods. He supplies one unit of labor during his two first periods of life, and is retired during the last period. The utility function of a generation t agent depends on his consumptions during the three periods, denoted by c_t, d_{t+1} and e_{t+2}, and is defined by:

$$U\left(c_t, d_{t+1}, e_{t+2}\right) = \gamma_1 \ln c_t + \gamma_2 \ln d_{t+1} + \gamma_3 \ln e_{t+2}$$

The parameters γ_i are positive and such that: $\gamma_1+\gamma_2+\gamma_3 = 1$. s_t is the amount of savings (the stock) held by the agent at the end of period t, and u_{t+1} this amount at the end of period $t+1$.

The production technology is Cobb-Douglas:

$$Y_t = F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$$

There is full depreciation of capital in one period. w_t is the wage in period t, R_t the factor of interest.

1. Write the 3 budget constraints of an agent of generation t, and the intertemporal budget constraint. Find the consumptions. Show that savings are equal to:

$$s_t = (1-\gamma_1)w_t - \gamma_1 \frac{w_{t+1}}{R_{t+1}}$$
$$u_{t+1} = \gamma_3 R_{t+1} w_t + \gamma_3 w_{t+1}$$

2. Write the firm program at period t. Give the optimality conditions. What is the total amount of labor supply at period t ? The variable k_t is defined as $k_t = K_t/(N_t + N_{t-1})$. Assuming an equilibrium on the labor market, give the equilibrium value of the wage w_t and of the interest factor R_t with respect to k_t.
3. Write the equation expressing the equilibrium of the capital market. Show that k_t follows a recurrence equation of order 2: an equation between k_{t+1}, k_t and k_{t-1}. Explain why the initial conditions need to make precise the values of both k_0 and k_{-1}.
4. A change of variable is introduced: $x_t = k_t/k_{t-1}^{\alpha}$. Is x_t a backward or forward variable? Show that the dynamics of x_t follows:

$$\left(2 + n + \gamma_1 \frac{1-\alpha}{\alpha}\right) x_{t+1} = (1-\gamma_1)(1-\alpha) + \frac{\gamma_3}{1+n}(1-\alpha)\left(1 + \frac{\alpha}{x_t}\right)$$

Study this dynamics with a graph. Show that x_t converges toward a stationary state with oscillations. The stationary state is denoted by x^* (it is not asked to find its value).
5. Write the program allowing to define the optimal stationary state of the economy. How is modified the standard golden rule? What is the value of x at the golden rule? Find the condition on the parameters of the economy ensuring that the stationary state of the competitive economy is optimal ?