

Chapter 4: Overlapping generations model and long run policies

A model of growth in which demography explicitly appears.

Designed for the study of growth, intergenerational transfers, public debt...

References:

D. Romer, Advanced macroeconomics, chapter 2B

O. J. Blanchard and S. Fisher: “Lectures in macroeconomics”, chapter 2.

D. De la Croix and P. Michel: “A Theory of Economic Growth: Dynamics and Policy in Overlapping generations”, Cambridge University Press, 2002.

1 The basic overlapping generations model

2 Intergenerational transfers and debt

1 The basic overlapping generations model

1.1 The demographic structure

In each period t , N_t persons are born, they live 2 periods.

First period is called youth, and second period old age.

During youth, agents are endowed with 1 unit of labor.
During old age, they are retired.

The number of households of each generation t grows at a constant rate $n \in (-1, +\infty)$:

$$N_t = (1 + n)N_{t-1}$$

In t , total population: $N_t + N_{t-1}$.

1.2 Budget constraints

In his first period of life, a generation t agent earns a wage w_t , which is allocated between consumption c_t and savings s_t :

$$c_t + s_t = w_t$$

During old age, the agent consumes an amount d_{t+1} equal to the proceed of savings. R_{t+1} is the return factor between t and $t + 1$.

$$d_{t+1} = R_{t+1}s_t$$

The intertemporal budget constraint is:

$$c_t + \frac{d_{t+1}}{R_{t+1}} = w_t$$

1.3 Consumer program

The utility function of an agent is:

$$U(c_t, d_{t+1})$$

Assumption 1: U strictly increasing and concave, twice continuously differentiable,

$$\lim_{c \rightarrow 0} U'_c = \lim_{d \rightarrow 0} U'_d = +\infty$$

The first-order conditions of the consumer program:

$$U'_{c_t} = R_{t+1} U'_{d_{t+1}}$$

Thus:

$$\begin{aligned} c_t &= c(w_t, R_{t+1}) \\ d_{t+1} &= d(w_t, R_{t+1}) \\ s_t &= s(w_t, R_{t+1}) \end{aligned}$$

For the first old people in $t = 0$:

$$d_0 = R_0 s_{-1} = \frac{R_0 K_0}{N_{-1}}$$

K_0 is the installed capital stock in $t = 0$, which is held by old agents.

Example: if $U(c, d) = (1-a) \ln c + a \ln d$, it is immediate that:

$$\begin{aligned}c_t &= (1-a)w_t \\d_{t+1} &= aw_t R_{t+1} \\s_t &= aw_t\end{aligned}$$

1.4 Firms and production

A representative firm exists at each period t , endowed with a neoclassical production technology.

$$Y_t = F(K_t, L_t)$$

K_t is the capital stock and L_t is the quantity of labor. We assume perfect competition.

F is homogenous of degree 1 (constant returns). Capital depreciates fully within the production period. We define f as: $f(k) \equiv F(k, 1)$.

Assumption 2: $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and for all $k > 0$, $f'(k) > 0$ and $f''(k) < 0$.

The program of the firm (profit maximization) is :

$$\max_{(K_t, L_t)} F(K_t, L_t) - R_t K_t - w_t L_t$$

The profit maximization gives:

$$w_t = F'_L(K_t, L_t) = f(K_t/L_t) - (K_t/L_t)f'(K_t/L_t)$$

$$\equiv w(K_t/L_t)$$

$$R_t = F'_K(K_t, L_t) = f'(K_t/L_t) \equiv R(K_t/L_t)$$

At the equilibrium, profit cancels out:

$$\begin{aligned} F(K_t, L_t) - R_t K_t - w_t L_t &= \\ F(K_t, L_t) - F'_K(K_t, L_t) K_t - F'_L(K_t, L_t) L_t &= 0 \end{aligned}$$

Example: if the production function is Cobb-Douglas $F(K, L) = K^\alpha L^{1-\alpha}$, w and R are such that:

$$\begin{aligned} w(K_t/L_t) &= (1 - \alpha)(K_t/L_t)^\alpha \\ R(K_t/L_t) &= \alpha(K_t/L_t)^{\alpha-1} \end{aligned}$$

1.5 Equilibrium with perfect foresight

In this economy, we have 3 markets : good, capital and labor. By the Walras' law, it is possible to consider only 2 markets.

Equilibrium on the labor market: $L_t = N_t$. We define the variable $k_t = K_t/N_t$. Thus we have the equilibrium value of prices:

$$\begin{aligned}w_t &= F'_L(k_t, 1) = f(k_t) - (k_t)f'(k_t) = w(k_t) \\R_t &= F'_K(k_t, 1) = f'(k_t) = R(k_t)\end{aligned}$$

Equilibrium on the capital market:

$$K_{t+1} = N_t s_t$$

the capital stock of period $t + 1$ is made by savings of all generation t agents. This equation can be written:

$$(1 + n)k_{t+1} = s_t$$

Equilibrium on the good market:

$$F(K_t, L_t) = N_t c_t + N_{t-1} d_t + K_{t+1}$$

With all the preceding relations, we find the dynamics of the economy:

$$(1 + n)k_{t+1} = s [w(k_t), R(k_{t+1})]$$

This is a dynamical equation of dimension 1.

Example: $U(c, d) = (1 - a) \ln c + a \ln d$, $F(K, L) = K^\alpha L^{1-\alpha}$. In this case, the dynamics is:

$$(1 + n)k_{t+1} = a(1 - \alpha)k_t^\alpha$$

1.6 Equilibrium with no assumption on expectations

If we do not make precise how agents make their expectations.

A young agent in t must expect the factor of interest $R_{t+1}^e = R(k_{t+1}^e)$.

Firms do not need expectations. Their behavior is static.

The dynamics become:

$$(1 + n)k_{t+1} = s \left[w(k_t), R(k_{t+1}^e) \right]$$

Example of expectations: static expectations, $k_{t+1}^e = k_t$.

1.7 Study of the dynamics in the example

Considering the dynamics

$$(1 + n)k_{t+1} = a(1 - \alpha)k_t^\alpha$$

there exist 2 steady states: 0 and k^* such that:

$$k^* = \left(\frac{a(1 - \alpha)}{1 + n} \right)^{\frac{1}{1-\alpha}}$$

k^* is globally stable.

Exercise: consider a CES Production function $F(K, L) = A (K^{-1} + L^{-1})^{-1}$

Prove that if $aA > 4(1 + n)$, there exists three steady states (0 and two positive steady states). There exists a poverty trap for k_0 too small.

1.8 Existence of an exogenous technical progress

We consider now agents whose productivity is affected by an exogenous technical progress h_t with $h_{t+1} = (1 + g)h_t$. We always assume that the utility function is log-linear.

The budget constraints become for a generation t individual with productivity h_t :

$$\begin{aligned}c_t + s_t &= w_t h_t \\d_{t+1} &= R_{t+1} s_t\end{aligned}$$

The savings function is:

$$s(w_t h_t, R_{t+1}) = a w_t h_t$$

The production function is now defined as:

$$F(K_t, H_t) \text{ with } H_t = N_t h_t$$

We set: $\kappa_t = K_t/H_t$. Equilibrium prices are:

$$w_t = F'_L(\kappa_t, 1) = f(\kappa_t) - (\kappa_t)f'(\kappa_t) = w(\kappa_t)$$

$$R_t = F'_K(\kappa_t, 1) = f'(\kappa_t) = R(\kappa_t)$$

w_t is the wage for one productive unit.

Finally the dynamics is obtained by setting:

$$K_{t+1} = N_t a h_t w_t$$

or:

$$\kappa_{t+1} = \frac{a w(\kappa_t)}{(1+n)(1+g)}$$

2 The optimal stationary state

What is the optimal stationary state ? It is the solution of the following program:

$$\begin{aligned} & \max_{(c,d,k)} U(c, d) \\ \text{s. t. } & c + \frac{d}{1+n} = f(k) - (1+n)k \end{aligned}$$

The optimal value of k is \hat{k} such that:

$$f'(\hat{k}) = 1 + n$$

(Golden rule). c and d satisfy the resource constraint and the arbitrage condition:

$$U'_c = (1+n)U'_d$$

If $k^* = \hat{k}$, the competitive stationary state corresponds to the golden rule.

If $k^* < \hat{k}$, the competitive stationary state of the economy is said to be in under-accumulation.

If $k^* > \hat{k}$, the competitive stationary state of the economy is said to be in over-accumulation.

Proposition 1 *A competitive stationary state in over-accumulation is not Pareto-optimal.*

Proposition 2 *A competitive equilibrium converging in under-accumulation is Pareto-optimal.*

3 Debt and intergenerational transfers

3.1 The economy with a social security system

All calculations are made under the assumptions: $U(c, d) = (1 - a) \ln c + a \ln d$, $F(K, L) = K^\alpha L^{1-\alpha}$.

3.1.1 The basic framework with PAYG pension system

The budget constraints become:

$$c_t + s_t = w_t(1 - \tau_t)$$

$$d_{t+1} = R_{t+1}s_t + p_{t+1}$$

The intertemporal budget constraint is:

$$c_t + \frac{d_{t+1}}{R_{t+1}} = w_t(1 - \tau_t) + \frac{p_{t+1}}{R_{t+1}}$$

The consumer program leads to:

$$s_t = s(w_t(1 - \tau_t), p_{t+1}, R_{t+1}) = aw_t(1 - \tau_t) - (1 - a)\frac{p_{t+1}}{R_{t+1}}$$

We assume that the budget constraint of the social security system is balanced at each period:

$$p_{t+1} = \tau_{t+1}(1 + n)w_{t+1}$$

The dynamics with the social security system is now:

$$k_{t+1} = \frac{a(1 - \alpha)(1 - \tau_t)k_t^\alpha}{(1 + n) \left[1 + (1 - a)\frac{1 - \alpha}{\alpha}\tau_{t+1} \right]}$$

The PAYG pension system has a negative impact of capital accumulation.

For a constant tax rate $\tau_t = \tau$, the dynamics leads to a new stationary state:

$$k^*(\tau) = \left\{ \frac{a(1 - \alpha)(1 - \tau)}{(1 + n) \left[1 + (1 - a)\frac{1 - \alpha}{\alpha}\tau \right]} \right\}^{\frac{1}{1 - \alpha}}$$

There exists a value of τ such that the competitive economy with a PAYG pension system reaches the optimal stationary state. This value is:

$$\hat{\tau} = a - \frac{\alpha}{1 - \alpha}$$

$\hat{\tau} > 0$ iff the economy without pension system is in over-accumulation.

3.1.2 A fully funded social security system

A fully funded system is neutral for capital accumulation. Its budget constraint is balanced if:

$$p_{t+1} = \tau_t w_t R_{t+1}$$

The intertemporal budget constraint of the consumer is the same as in the economy without pension system:

$$c_t + \frac{d_{t+1}}{R_{t+1}} = w_t$$

Total private savings are:

$$N_t s(w_t, R_{t+1}) - N_t \tau_t w_t$$

Savings of the pension system are:

$$N_t \tau_t w_t$$

Therefore, total savings remain the same.

3.1.3 The problem of the transition from a PAYG to a fully funded system

3.2 Public debt

The government has two instruments: a tax (transfer) on young agents and debt. Public debt has the same return as capital: R_{t+1} (arbitrage condition).

The budget constraints become:

$$c_t + s_t = w_t - \tau_t$$

$$d_{t+1} = R_{t+1}s_t$$

τ_t is a lump sum tax.

The consumer program leads to:

$$U'_{c_t} = R_{t+1}U'_{d_{t+1}}$$

which leads to:

$$s_t = s(w_t - \tau_t R_{t+1})$$

Government budget constraint:

$$B_t = R_t B_{t-1} + G_t - N_t \tau_t$$

B_t is the amount of debt issued at the end of period t .

G_t is the amount of public spending..

The level of debt per young agent evolves as:

$$b_t = \frac{R_t}{1+n} b_{t-1} + g_t - \tau_t$$

with $g_t = G_t/N_t$.

The equilibrium of the capital market implies:

$$N_t s_t = K_{t+1} + B_t$$

or

$$s_t = (1 + n)k_{t+1} + b_t$$

The constraint $k_{t+1} > 0$ implies a constraint on debt emission.

We consider the dynamics obtained for a constant level of debt per young agent: $b_t = b$.

$$\tau_t = b \left[\frac{R(k_t)}{1 + n} - 1 \right]$$

$$(1+n)k_{t+1} + b = s \left[w(k_t) - b \left[\frac{R(k_t)}{1 + n} - 1 \right], R(k_{t+1}) \right]$$

$$(1 + n)k_{t+1} = a(1 - \alpha)k_t^\alpha - ab \frac{\alpha k_t^{\alpha-1}}{1 + n} - (1 - a)b$$

If b not too high, there exist 2 stationary states \underline{k}_b and \bar{k}_b with $\underline{k}_b < \bar{k}_b$. \underline{k}_b is unstable and \bar{k}_b is stable. \underline{k}_b increases with b and \bar{k}_b decreases with b . If $k_0 < \underline{k}_b$, k_t becomes negative in a finite time, which is impossible. It is the case of a underdevelopment threshold.

There exists a limit value of $b = b_{\max}$ such that $\underline{k}_b = \bar{k}_b = \bar{k}$. In this case, the stationary state is unstable at left and stable at right.

For a given value of k_0 , there exists a maximal value of debt by young agent that is sustainable. If $k_0 \geq \bar{k}$, this value is b_{\max} . If $k_0 < \bar{k}$, this value is b such that $\underline{k}_b = k_0$.