Chapter 3: Economic growth with endogenous saving behavior: The Ramsey-Cass-Koopmans model

1 The Ramsey problem

Optimal growth problem introduced by Ramsey:

$$\begin{aligned} & \max \sum_{t=0}^{+\infty} \beta^t u(c_t) \\ \text{s. t. } N_t c_t + K_{t+1} &= F(K_t, N_t) + (1-\delta) K_t \\ & K_0 \text{ given.} \end{aligned}$$

or

$$\begin{aligned} \text{s. t. } c_t + (1+n) k_{t+1} &= f(k_t) + (1-\delta) k_t \\ & k_0 \text{ given.} \end{aligned}$$

Lagragien:

$$\mathcal{L} = u(c_t) + \beta \frac{\lambda_{t+1}}{1+n} \left(f(k_t) + (1-\delta)k_t - c_t \right) - \lambda_t k_t$$

The first order condition:

$$u'(c_t) = \beta \frac{f'(k_{t+1}) + 1 - \delta}{1+n} u'(c_{t+1})$$

A limit condition:

$$\lim_{t \to \infty} \beta^t \lambda_t k_t = 0$$

with λ_t the implicit price of capital.

We assume the existence of a stationary state $\bar{k}$ such that:

$$f'(\bar{k}) + 1 - \delta = (1+n) / \beta$$

The corresponding value of consumption is:

$$\bar{c} = f(\bar{k}) + (1-\delta)\bar{k} - (1+n)\bar{k}$$

Comparison with the golden rule $\left(\hat{c}, \hat{k}\right)$:

$$\begin{aligned} f'(\bar{k}) &= \delta + n \\ \hat{c} &= f(\hat{k}) - (\delta + n)\hat{k} \end{aligned}$$

Phase diagram.

Explicit resolution in the case: $u(c) = \ln c$, $f(k) = k^{\alpha}$, $\delta = 1$.

Change of variable: $x_t = \lambda_t k_t$, gives $\forall t$, $x_t = \alpha/(1 - \alpha\beta)$.

Results:

$$\begin{aligned} c_t &= (1 - \alpha\beta)k_t^{\alpha} \\ (1+n)k_{t+1} &= \alpha\beta k_t^{\alpha} \end{aligned}$$

2 The decentralized model

We compare the centralized (social optimum solution) with the decentralized one.

The representative consumer:

$$
\begin{aligned}
& \max \sum_{t=0}^{+\infty} \beta^t u(c_t) \\
a_{t+1} &= \frac{1}{1+n}\left[(1+r_t)a_t + w_t - c_t\right] \\
& a_0 \text{ given} \\
\lim_{t\to\infty} \rho_t N_{t+1} a_{t+1} &\geq 0
\end{aligned}
$$

Solution:

$$u'(c_t) = \beta \frac{1+r_{t+1}}{1+n} u'(c_{t+1})$$

$$\lim_{t\to\infty} \rho_t N_{t+1} a_{t+1} = 0$$

The productive sector:

$$
\begin{aligned}
f'(K_t/L_t) &= (r_t + \delta) \\
f(K_t/L_t) - K_t/L_t f'(K_t/L_t) &= w_t
\end{aligned}
$$

Equilibrium conditions:

Labor market: $L_t = N_t$ or $K_t/L_t = k_t$.

Capital market: $A_t = K_t$ or $a_t = k_t$.

Consequences: The budget constraint of the consumer is equivalent to the resource constraint.

The FOC is the same as for the centralized economy.

The limit condition:

For the social planner,

$$\lambda_t = \frac{\beta \lambda_{t+1} \left(f'(k_{t+1}) + 1 - \delta \right)}{1 + n}$$

or

$$\lambda_t = \lambda_0 \frac{(1+n)^t}{\beta^t} \rho_t$$

Therefore, limit conditions are equivalent.

Equivalence of the competitive equilibrium with one representative agent and the social planer solution. (First Welfare Theorem).

3 Introduction of a technical progress

Assume now that production is given by $F(K_t, A_t N_t)$ with $A_t = A_0(1+a)^t$.

We define:

$$\begin{aligned}\tilde{c}_t &= c_t/A_t \\ \tilde{k}_t &= k_t/A_t\end{aligned}$$

The resource constraint is now:

$$\tilde{c}_t + (1+n)(1+a)\tilde{k}_{t+1} = f(\tilde{k}_t) + (1-\delta)\tilde{k}_t$$

and the objective function:

$$\sum_{t=0}^{T} \beta^t u(A_t \tilde{c}_t)$$

Assume that u is a CES function:

$$u(c) = \frac{c^{1-1/\sigma}}{1-1/\sigma}$$

The objective function can be written:

$$\sum_{t=0}^{T} \beta^t A_0^{1-1/\sigma} (1+a)^{t(1-1/\sigma)} \frac{\tilde{c}_t^{1-1/\sigma}}{1-1/\sigma}$$

Therefore, the problem is the same as the basic one if we take:

$$\begin{aligned} \beta' &\rightarrow \beta(1+a)^{1-1/\sigma} \\ (1+n)' &\rightarrow (1+n)(1+a) \end{aligned}$$

The golden rule becomes:

$$f'(\bar{k}) + 1 - \delta = (1+n)\,(1+a)^{1/\sigma}/\beta$$