Overlapping Generations and Growth

B. Wigniolle

Overlapping Generations and Growth

A model of growth where demography explicitly appears.

Designed for the study of growth, intergenerational transfers, public debt...

References:

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1 The basic overlapping generations model

1.1 The demographic structure

In each period t, N_t persons are born, they live 2 periods.

First period is called youth, and second period old age.

During youth, agents are endowed with 1 unit of labor. During old age, they are retired.

The number of households of each generation t grows at a constant rate $n \in (-1, +\infty)$:

$$N_t = (1+n)N_{t-1}$$

In t, total population: $N_t + N_{t-1}$.

1.2 Budget constraints

In his first period of life, a generation t agent earns a wage w_t, which is allocated between consumption c_t and savings s_t :

$$c_t + s_t = w_t$$

During old age, the agent consumes an amount d_{t+1} equal to the proceed of savings. R_{t+1} is the return factor between t and $t+1$.

$$d_{t+1} = R_{t+1} s_t$$

The intertemporal budget constraint is:

$$c_t + \frac{d_{t+1}}{R_{t+1}} = w_t$$

1.3 Consumer program

The utility function of an agent is:

$$U\left(c_t, d_{t+1}\right)$$

Assumption 1:U strictly increasing and concave, twice continuously differentiable,

$$\lim_{c\to 0} U'_c = \lim_{d\to 0} U'_d = +\infty$$

The first-order conditions of the consumer program:

$$U'_{c_t} = R_{t+1} U'_{d_{t+1}}$$

Thus:

$$\begin{aligned} c_t &= c(w_t, R_{t+1}) \\ d_{t+1} &= d(w_t, R_{t+1}) \\ s_t &= s(w_t, R_{t+1}) \end{aligned}$$

For the first old people in $t = 0$:

$$d_0 = R_0 s_{-1} = \frac{R_0 K_0}{N_{-1}}$$

K_0 is the installed capital stock in $t = 0$, which is held by old agents.

1.4 Firms and production

A representative firm exists at each period t, endowed with a neoclassical production technology.

$$Y_t = F(K_t, L_t)$$

K_t is the capital stock and L_t is the quantity of labor. We assume perfect competition.

F is homogenous of degree 1 (constant returns). Capital depreciates fully within the production period. We define f as: $f(k) \equiv F(k, 1)$.

Assumption 2: $f : \mathbb{R}_+ \to \mathbb{R}_+$, and for all $k > 0$, $f'(k) > 0$ and $f''(k) < 0$.

The program of the firm (profit maximization) is :

$$\max_{(K_t, L_t)} F(K_t, L_t) - R_t K_t - w_t L_t$$

The profit maximization gives:

$$\begin{aligned} w_t &= F'_L(K_t, L_t) = f(K_t/L_t) - (K_t/L_t) f'(K_t/L_t) \\ &\equiv w(K_t/L_t) \\ R_t &= F'_K(K_t, L_t) = f'(K_t/L_t) \equiv R(K_t/L_t) \end{aligned}$$

At the equilibrium, profit cancels out:

$$\begin{aligned} F(K_t, L_t) - R_t K_t - w_t L_t &= \\ F(K_t, L_t) - F'_K(K_t, L_t) K_t - F'_L(K_t, L_t) L_t &= 0 \end{aligned}$$

1.5 Equilibrium with perfect foresight

In this economy, we have 3 markets : good, capital and labor. By the Walras' law, it is possible to consider only 2 markets.

Equilibrium on the labor market: $L_t = N_t$. We define the variable $k_t = K_t/N_t$. Thus we have the equilibrium value of prices:

$$\begin{aligned} w_t &= F'_L(k_t, 1) = f(k_t) - (k_t)f'(k_t) = w(k_t) \\ R_t &= F'_K(k_t, 1) = f'(k_t) = R(k_t) \end{aligned}$$

Equilibrium on the capital market:

$$K_{t+1} = N_t s_t$$

the capital stock of period $t+1$ is made by savings of all generation t agents. This equation can be written:

$$(1+n)k_{t+1} = s_t$$

Equilibrium on the good market:

$$F(K_t, L_t) = N_t c_t + N_{t-1} d_t + K_{t+1}$$

With all the preceding relations, we find the dynamics of the economy:

$$(1+n)k_{t+1} = s\left[w(k_t), R(k_{t+1})\right]$$

This is a dynamical equation of dimension 1.

Example: $U(c,d) = (1-a)\ln c + a \ln d$, $F(K,L) = K^\alpha L^{1-\alpha}$.

1.6 Equilibrium with no assumption on expectations

If we do not precise how agents make their expectations.

A young agent in t must expect the factor of interest $R^e_{t+1} = R(k^e_{t+1})$.

Firms do not need expectations. Their behavior is static.

The dynamics become:

$$(1+n)k_{t+1} = s\left[w(k_t), R(k^e_{t+1})\right]$$

Example of expectations: static expectations, $k^e_{t+1} = k_t$.

1.7 Existence and uniqueness of the competitive dynamics with perfect foresights

Considering the dynamics with perfect foresights,

$$(1+n)k_{t+1} = s\left[w(k_t), R(k_{t+1})\right] \tag{1}$$

we study the following problem:

$\forall\ k_t > 0$, Does it exist k_{t+1} such that (1) is satisfied ?

Is k_{t+1} exists, is it unique ?

When these 2 properties hold, we say that the dynamics is well defined.

Assumption 3: $s(w, R)$ is non-decreasing with respect to R ($s_R \geq 0$). c and d are normal goods.

The substitution effect dominates the revenue effect.

Proposition 1 *Under the preceding assumptions, the dynamics is well defined.*

Corollary 2 *When the preceding assumption is satisfied, the dynamics is monotonic.*

k_{t+1} is an increasing function of k_t. Either (k_t) is an increasing sequence, or a decreasing one.

1.8 Existence of steady states and local dynamics

A steady state k^* is a value such that:

$$(1+n)k^* = s\left[w(k^*), R(k^*)\right]$$

Proposition 3 *There exists no more than one positive steady state k^* of the dynamics with* *perfect foresight when:*

$$\forall k > 0,\ s'_w w'(k) + s'_R R'(k) < \frac{s\left[w(k), R(k)\right]}{k}$$

Such a steady state exists if and only if

$$\lim_{k \to 0} \frac{s\left[w(k), R(k)\right]}{k} > 1 + n$$

Remark 1 *A necessary condition for that the last condition holds is that*

$$\lim_{k \to 0} \frac{w(k)}{k} > 1 + n$$

The dynamics with perfect foresights is written:

$$k_{t+1} = g(k_t)$$

g is such that:

$$g'(k) = \frac{s'_w w'(k)}{1 + n - s'_R R'(k)}$$

k^* is locally stable if:

$$|g'(k)| < 1$$

When assumption 3 holds, we have

$$g'(k) > 0$$

and the preceding condition becomes:

$$g'(k) < 1$$

1.9 Global dynamics

We assume that all the preceding assumptions are satisfied. We know that the dynamics is monotonic.

Starting from k_0, if the dynamics is decreasing, either it converges toward a steady state ≥ 0 or the economy disappears in a finite time.

Indeed, (k_t) is decreasing and bounded by 0.

If the dynamics is increasing, it necessarily converges toward a steady state > 0 : the dynamics is bounded.

Proof:

The proof lies on the property that $\lim\limits_{k \to +\infty} w(k)/k = 0$.

1.10 Examples

1.10.1 Utility and production functions

are Cobb-Douglas

$F(K, L) = K^{\alpha} L^{1-\alpha}$ and $U(c, d) = (1-a) \ln c + a \ln d$.

1.10.2 A CES Production function

$F(K, L) = A \left(K^{-1} + L^{-1} \right)^{-1}$

If $aA > 4(1+n)$, there exists three steady states (0 and two positive steady states). There exists a poverty trap for k_0 to small.

1.11 Extensions

1.11.1 The case of heterogenous agents

When agents are endowed with homothetic preferences, they can be easily aggregated..

Preferences are said homothetic when the indifference curves are homothetic with respect to the origin. Such preferences can be represented by a utility function $U(c, d)$ which is homogenous of degree 1.

When preferences are homothetic, the savings function $s(w, R)$ is linear with respect to

w and satisfies:

$$s(w, R) = \zeta(R)w$$

and

$$\zeta(R) = \max_{\zeta} U(1 - \zeta, R\zeta)$$

We assume that people differ by their labor productivity level h. The proportion of agents with the same productivity level h is assumed to be constant among the different generations, and given by a density function $\mu(h)$.

We have

$$\int_{0}^{+\infty} d\mu = 1$$

and

$$N_t \int_{h_1}^{h_2} d\mu$$

is the mass of generation t agents endowed with a productivity level between h_1 and h_2.

We denote by H the average level of productivity:

$$H = \int_{0}^{+\infty} h d\mu$$

Each agent has an homothetic utility function $U(c,d)$.

The budget constraints become for an individual with productivity h :

$$\begin{aligned} c_t + s_t &= w_t h \\ d_{t+1} &= R_{t+1} s_t \end{aligned}$$

The savings function is:

$$s\left(w_t h, R_{t+1}\right) = w_t h \zeta(R_{t+1})$$

Total savings are:

$$N_t \int_{0}^{+\infty} w_t h \zeta(R_{t+1}) d\mu = N_t w_t H \zeta(R_{t+1})$$

The production function is now defined as:

$F(K_t, H_t)$ with $H_t = N_t H$

We set: $\kappa_t = K_t / H_t$. Equilibrium prices are:

$$\begin{aligned} w_t &= F'_L(\kappa_t, 1) = f(\kappa_t) - (\kappa_t) f'(\kappa_t) = w(\kappa_t) \\ R_t &= F'_K(\kappa_t, 1) = f'(\kappa_t) = R(\kappa_t) \end{aligned}$$

w_t is the wage for one productive unit.

Finally the dynamics is obtained by setting:

$$K_{t+1} = N_t w_t H \zeta(R_{t+1})$$

or:

$$\kappa_{t+1} = \frac{w(\kappa_t)\zeta\left[R(\kappa_{t+1})\right]}{1+n}$$

1.11.2 Existence of an exogenous technical progress

We consider now homogenous agents, whose productivity is affected by an exogenous technical progress h_t with $h_{t+1} = (1+g)h_t$. We always assume that preferences are homothetic.

The budget constraints become for a generation t individual with productivity h_t :

$$\begin{aligned} c_t + s_t &= w_t h_t \\ d_{t+1} &= R_{t+1} s_t \end{aligned}$$

The savings function is:

$$s\left(w_t h_t, R_{t+1}\right) = w_t h_t \zeta(R_{t+1})$$

The production function is now defined as:

$F(K_t, H_t)$ with $H_t = N_t h_t$

We set: $\kappa_t = K_t/H_t$. Equilibrium prices are:

$$w_t = F'_L(\kappa_t, 1) = f(\kappa_t) - (\kappa_t)f'(\kappa_t) = w(\kappa_t)$$

$$R_t = F'_K(\kappa_t, 1) = f'(\kappa_t) = R(\kappa_t)$$

w_t is the wage for one productive unit.

Finally the dynamics is obtained by setting:

$$K_{t+1} = N_t h_t w_t \zeta(R_{t+1})$$

or:

$$\kappa_{t+1} = \frac{w(\kappa_t)\zeta\left[R(\kappa_{t+1})\right]}{(1+n)(1+g)}$$

1.12 Optimality

1.12.1 The optimal stationary state

What is the optimal stationary state ? It is the solution of the following program:

$$\max_{(c,d,k)} U(c,d)$$

$$\text{s. t. } c+\frac{d}{1+n} = f(k)-(1+n)k$$

The optimal value of k is $\widehat{k}$ such that:

$$f'(\widehat{k}) = 1+n$$

(Golden rule). c and d satisfy the resource constraint and the arbitrage condition:

$$U_c' = (1+n)U_d'$$

If $k^* = \widehat{k}$, the competitive stationary state corresponds to the golden rule.

If $k^* < \widehat{k}$, the competitive stationary state of the economy is said to be in under-accumulation.

If $k^* > \widehat{k}$, the competitive stationary state of the economy is said to be in over-accumulation.

Proposition 4 *A competitive stationary state* *in over-accumulation is not Pareto-optimal.*

Proposition 5 *A competitive equilibrium converging in under-accumulation is Pareto-optimal.*

1.12.2 The planning problem

We assume that the social planner has the following objective:

$$\sum_{t=-1}^{+\infty} \gamma^t U(c_t, d_{t+1})$$

which is maximized under the resource constraint:

$$c_t + \frac{d_t}{1+n} + (1+n)k_{t+1} = f(k_t)$$

c_{-1} and k_0 are given in this program. $\gamma < 1$ is the discount parameter of the social planner.

2 Debt and intergenerational transfers

2.1 A PAYG social security system

2.1.1 The basic framework

The budget constraints become:

$$c_t + s_t = w_t(1 - \tau_t)$$

$$d_{t+1} = R_{t+1}s_t + p_{t+1}$$

The intertemporal budget constraint is:

$$c_t + \frac{d_{t+1}}{R_{t+1}} = w_t(1 - \tau_t) + \frac{p_{t+1}}{R_{t+1}}$$

The consumer program leads to:

$$s_t = s(w_t(1 - \tau_t), p_{t+1}, R_{t+1})$$

Under the preceding assumptions, s decreases with τ_t and with p_{t+1}.

We assume that the budget constraint of the social security system is balanced at each period:

$$p_{t+1} = \tau_{t+1}(1 + n)w_{t+1}$$

The dynamics with the social security system is now:

$$k_{t+1} = \frac{s\left[w(k_t)(1-\tau_t), \tau_{t+1}(1+n)w(k_{t+1}), R(k_{t+1})\right]}{1+n}$$

Example: a Cobb-Douglas economy with a constant tax rate $\tau_t = \tau$.

Proposition 6 *Considering the optimal trajectory determined by a social planner* $\left(\overline{c}_t, \overline{d}_t, \overline{k}_t\right)_{t\geq 0}$, *this trajectory can be decentralized by an appropriate choice of* τ_t.

2.1.2 A fully funded social security system

A fully funded system is neutral for capital accumulation.

2.1.3 The problem of the transition from a PAYG to a fully funded system

2.2 Public debt

2.2.1 The basic framework

The government has three instruments: taxes (transfers) on young and old people and debt.

Public debt has the same return as capital: R_{t+1} (arbitrage condition).

The budget constraints become:

$$c_t + s_t = w_t - \tau_t^1$$

$$d_{t+1} = R_{t+1}s_t - \tau_t^2$$

τ_t^1 and τ_t^2 are lump sum taxes.

The consumer program leads to:

$$U'_{c_t} = R_{t+1}U'_{d_{t+1}}$$

which leads to:

$$s_t = s(w_t - \tau_t^1, -\tau_t^2, R_{t+1})$$

Under the preceding assumptions, s decreases with τ_t^1 and with $-\tau_t^2$.

Government budget constraint:

$$B_t = R_t B_{t-1} + G_t - N_t\tau_t^1 - N_{t-1}\tau_t^2$$

B_t is the amount of debt issued at the end of period t. G_t is the amount of public spending..

The level of debt per young agent evolves as:

$$b_t = \frac{R_t}{1+n}b_{t-1} + g_t - \tau_t^1 - \frac{\tau_t^2}{1+n}$$

with $g_t = G_t/N_t$.

2.2.2 The competitive equilibrium with

perfect foresights

For given initial values of K_0, B_{-1}, $s_{-1} = (K_0 + B_{-1})/N_{-1}$, and for a sequence $(G_t, \tau_t^1, \tau_t^2)_{t \geq 0}$, a competitive equilibrium with perfect foresights is a sequence $(K_{t+1}, B_t, c_t, d_t, s_t, w_t, R_t)_{t \geq 0}$ with $K_{t+1} > 0$, $c_t > 0$, $d_t > 0$, $w_t > 0$, $R_t > 0$ which satisfies the following conditions:

(1) optimal choice for consumers under the budget constraints:

$$c_t + s_t = w_t - \tau_t^1$$

$$d_{t+1} = R_{t+1} s_t - \tau_t^2$$

$$U'_{c_t} = R_{t+1} U'_{d_{t+1}}$$

(1) conditions of profit maximization for firms
$F_K(K_t, N_t) = R_t$ and $F_L(K_t, N_t) = w_t$

(2) The government budget constraint:
$B_t = R_t B_{t-1} + G_t - N_t \tau_t^1 - N_{t-1} \tau_t^2$

(3) Equilibrium of the capital market, or of the

good market:

$$N_t s_t = K_{t+1} + B_t$$

or

$$F(K_t, N_t) = K_{t+1} + G_t + N_t c_t + N_{t-1} d_t$$

The constraint $K_{t+1} > 0$ implies a constraint on debt emission.

2.2.3 Debt emission when the government has three instruments

Definition: *A sequence of positive quantities* $(c_t, d_t, G_t, K_t)_{t \geq 0}$ *is said to be realizable, if it satisfies the resource constraint of the economy and the arbitrage condition of the consumers, i.e. :*

1) $F(K_t, N_t) = K_{t+1} + G_t + N_t c_t + N_{t-1} d_t$
2) $U_c(c_t, d_{t+1}) = F_K(K_{t+1}, N_{t+1}) U_d(c_t, d_{t+1})$

Then, we have the result:

Proposition 7 *For all realizable trajectory and for all sequence of public debt* $(B_t)_{t \geq -1}$, *there exists one sequence* $(\tau_t^1, \tau_t^2)_{t \geq 0}$ *for which this trajectory is a competitive equilibrium asso-*

ciated with a sequence of economic policies $(B_{t-1}, \tau_t^1, \tau_t^2)_{t\geq 0}$.

One instrument is redundant.

2.2.4 Debt emission when the government has two instruments

We assume that $\tau_t^2 = 0$. We note $\tau_t^1 = \tau_t$.

We consider the dynamics obtained for a constant level of debt per young agent: $b_t = b$.

$$\tau_t = b\left[\frac{R(k_t)}{1+n} - 1\right]$$

$$(1+n)k_{t+1} + b = s\left[w(k_t) - b\left[\frac{R(k_t)}{1+n} - 1\right], R(k_{t+1})\right]$$

Example: the Cobb-Douglas Case

General case: the impact of a marginal debt on capital accumulation at the stationary state.

2.2.5 Debt emission without any other instrument: the case of the Ponzy Game.

Study of the Cobb-Douglas example.

3 Altruism

3.1 Descendant altruism

3.1.1 The model (Barro 1974, JPE)

Parents may influence the utility of their children by transfers (bequest).

The new budget constraints are:

$$c_t + s_t = w_t + b_t$$

$$d_{t+1} + (1+n)b_{t+1} = R_{t+1}s_t$$

For a generation t agent, b_t is given, but b_{t+1} is the bequest that he can give.

We assume the existence of a constraint: $b_{t+1} \geq 0$.

The utility of an agent depends on the utility of his descendants:

$$V_t = U(c_t, d_{t+1}) + \beta V_{t+1}$$

Developing forward this equation, we obtain:

$$V_t = \sum_{s=0}^{+\infty} \beta^s U(c_{t+s}, d_{t+1+s})$$

3.1.2 First order conditions

$$\begin{aligned} U'_{c_t} &= R_{t+1} U'_{d_{t+1}} \\ \beta U'_{c_{t+1}} &\leq (1+n) U'_{d_{t+1}} \text{ with equality if } b_{t+1} > 0 \end{aligned}$$

3.1.3 Condition for having operative bequests

At the stationary state,

$$b > 0 \Leftrightarrow s\left[w\left(\widetilde{k}\right), R\left(\widetilde{k}\right)\right] < (1+n)\widetilde{k}$$

with $\widetilde{k}$ the modified golden rule such that

$$R\left(\widetilde{k}\right) = \frac{1+n}{\beta}$$

3.1.4 Ricardian Neutrality

3.2 Ascendant altruism

3.2.1 The model

Children may influence the utility of their parent by transfers (gift).

The new budget constraints are:

$$c_t + s_t + g_t = w_t$$
$$d_{t+1} = R_{t+1}s_t + (1+n)g_{t+1}$$

For a generation t agent, g_{t+1} is given, but g_t is the gift that he can give.

We assume the existence of a constraint: $g_t \geq 0$.

The utility of an agent depends on the utility of his parent:

$$V_t = U(c_t, d_{t+1}) + \theta V_{t-1}$$

Developing backward this equation, we obtain:

$$V_t = \sum_{s=0}^{+\infty} \theta^s u(c_{t-s}, d_{t+1-s})$$

3.2.2 First order conditions

$$\begin{aligned} U'_{c_t} &= R_{t+1}U'_{d_{t+1}} \\ \theta(1+n)U'_{d_t} &\leq U'_{c_t} \text{ with equality if } g_t > 0 \end{aligned}$$

3.2.3 Condition for having operative gifts

At the stationary state,

$$g > 0 \Leftrightarrow s\left[w\left(\underset{\sim}{k}\right), R\left(\underset{\sim}{k}\right)\right] > (1+n)\,\underset{\sim}{k}$$

with $\underset{\sim}{k}$ such that

$$R\left(\underset{\sim}{k}\right) = (1+n)\theta$$

3.2.4 Ricardian Neutrality

3.3 Two-sided altruism

Different formulations exist:

Kimball (1987, JME):

$$V_t = U(c_t, d_{t+1}) + \beta V_{t+1} + \theta V_{t-1}$$

Burbidge (1983, AER):

$$V_t = \sum_{s=-1}^{+\infty} \beta^s U(c_{t+s}, d_{t+1+s})$$

The budget constraints are:

$$c_t + s_t + g_t = w_t + b_t$$

$$d_{t+1} + (1+n)b_t = R_{t+1}s_t + (1+n)g_{t+1}$$

A more general formulation, Abel (1987, AER)

$$V_t = \theta U(c_{t-1}, d_t) + \sum_{s=0}^{+\infty} \beta^s U(c_{t+s}, d_{t+1+s})$$

4 Endogenous growth

4.1 Externalities and growth

4.1.1 A model with an externality *à la Romer* (1986)

We consider the basic OLG model with a homothetic utility function, and a production function given by:

$$F(K_t, h_t L_t)$$

h_t is an endogenous technical progress, which is equal *ex-post* to $K_t/N_t = k_t$: the aggregate stock of capital per agent is assumed to be a measure of the knowledge of labor.

h_t is an externality.

At equilibrium, we have:

$$w_t = k_t F'_L(K_t, K_t) = k_t F'_L(1, 1)$$

$$R_t = F'_K(1,1)$$

The dynamics is given by:

$$(1+n)k_{t+1} = s(w_t, R_{t+1}) = k_t F'_L(1,1)\zeta(F'_K(1,1))$$

We obtain a constant growth rate of the capital stock per young agent k_t.

It is possible to use another close assumption:

$$h_t = K_t$$

with a constant population: $N_t = N$.

4.1.2 Barro's (1990) assumption

Barro introduces an assumption which can explain the existence of the externality h_t.

We consider the production function:

$$F(K_t, L_t, G_t) = K_t^{\alpha} L_t^{1-\alpha} G_t^{\gamma}$$

G_t is a public good which is financed by a tax on wages τ. This public good is an externality for private producers. It represents public infrastructures such as roads, public education, communication networks, etc...

We assume that population is constant $N_t = N$, and that the utility function is log-linear.

For $\gamma = 1-\alpha$, capital stock and revenues increase at a constant rate. This rate is maximum for $\tau = 1 - \alpha$.

4.2 Human capital accumulation and growth

4.2.1 The model of Glomm and Ravikumar (1992).

We consider now one period of learning before the period of work. To simplify the analysis, we neglect the period of retirement (without loss of generality) which would be a third period of life. Agents only consume during their second period of life.

Population is constant: one agent gives birth to one child.

The utility of an agent is now:

$$\alpha \ln c_t + (1 - \alpha) \ln l_t + \beta \ln h_{t+1}$$

c_t is consumption at the second period, and l_t is leisure.

h_{t+1} is le human capital level of the child.

h_{t+1} results from a technology of human capital accumulation:

$$h_{t+1} = \theta h_t^{\delta} e_t^{1-\delta}$$

h_t is the human capital level of the parent. e_t are education expenses that are made on the child.

Either e_t are financed by the parent, or by a public system of education.

The production technology is:

$$F(H_t) = H_t$$

where H_t is the total stock of human capital. Thus the competitive wage of an individual endowed with a human capital level h_t is h_t.

Finally, the budget constraint of a generation t agent is:

$$c_t + e_t = (1 - l_t)h_t$$

in a private regime of education, and

$$c_t = (1 - \tau_t)(1 - l_t)h_t$$

in a public system.

4.2.2 Dynamics of the economy with a private system

4.2.3 Dynamics of the economy with a public system

The tax is determined by a vote.

The public system is less efficient, because it creates an externality. But, when population is heterogenous, it leads to less inequality.

We first consider the case of an economy with homogenous agents.

5 Endogenous fertility

5.1 A simple model

At each date t, N_t identical agents are born. Their utility function:

$$U(c_t, d_{t+1}, m_t)$$

c_t and d_{t+1} are consumptions during youth and old age, and m_t is the number of children.

The number of generation $t+1$ agents:

$$N_{t+1} = m_t N_t$$

Assumption 1 : U twice continuously differentiable, strictly concave, increasing, satisfies the Inada conditions:

$$\lim_{c\to 0} U'_c = \lim_{d\to 0} U'_d = \lim_{m\to 0} U'_m = +\infty$$

The first period budget constraint :

$$c_t + \phi m_t + s_t = w_t$$

The second period budget constraint is:

$$d_{t+1} = R_{t+1} s_t$$

The lifetime budget constraint:

$$c_t + \frac{d_{t+1}}{R_{t+1}} + \phi m_t = w_t$$

The first-order conditions of the consumer program:

$$U_{c_t} = R_{t+1} U_{d_{t+1}} = \frac{U_{m_t}}{\phi}$$

Thus:

$$\begin{aligned} c_t &= c(w_t, R_{t+1}) \\ m_t &= m(w_t, R_{t+1}) \\ s_t &= s(w_t, R_{t+1}) \\ d_{t+1} &= d(w_t, R_{t+1}) \end{aligned}$$

At date $t = 0$, N_{-1} old agents consume: $d_0 = R_0 s_{-1}$. N_{-1}, s_{-1} and N_0 are given.

5.2 The intertemporel equilibrium

5.2.1 Equilibrium characterization

Definition 1: *Starting from positive initial conditions* N_{-1}, N_0, K_0 *and* $d_0 = R_0 K_0 / N_{-1}$, *an intertemporal equilibrium is a sequence* $(K_t$, N_t, c_t, d_t, $m_t)_{t \geq 0}$ *satisfying:*

$$c_t + \frac{d_{t+1}}{R_{t+1}} + \phi m_t = w_t$$

$$\begin{aligned} U_{c_t} &= R_{t+1} U_{d_{t+1}} = \frac{U_{m_t}}{\phi} \\ N_{t+1} &= m_t N_t \\ F(K_t, N_t) &= N_t c_t + N_{t-1} d_t + K_{t+1} + N_t \phi m_t \\ \text{with } R_t &= R(k_t) \text{ and } w_t = w(k_t) \\ \text{where } &: \;\; k_t = K_t / N_t \end{aligned}$$

Proposition 1 (Characterization of the equilibrium): *An intertemporal equilibrium is characterized by a sequence* $(k_t)_{t\geq 0}$ *with* $k_t > 0$ *such that:* $k_0 = K_0/N_0$ *is given, and* $\forall t \geq 0$,
$m\left[w(k_t), R(k_{t+1})\right] k_{t+1} - s\left[w(k_t), R(k_{t+1})\right] = 0$

5.2.2 Sufficient conditions for existence and uniqueness of the intertemporal equilibrium

We define the function Δ as:

$$\Delta(w, k) = m(w, f'(k))k - s(w, f'(k))$$

The dynamics becomes:

$$\Delta\left(w(k_t), k_{t+1}\right) = 0$$

Assumption 3: c and m are gross substitutes to d, that is $c'_R \leq 0$ and $m'_R \leq 0$.

As a consequence, we have

$$s'_R = -c'_R - \phi m'_R \geq 0$$

Example 1 *:*

$$U(c, d, m) = u_1(c) + u_2(d) + v(m)$$

with $u_1' > 0$, $u_2' > 0$, $v' > 0$, $u_1'' < 0$, $u_2'' < 0$, $v'' < 0$. *Assumption 3 is satisfied iff*

$$du_2''(d)/u_2'(d) \geq -1$$

Proposition 2: *Under assumptions 1, 2 and 3,* $\forall k_0 > 0$, *there exists a unique intertemporal equilibrium* $(k_t)_{t \geq 0}$ *starting from a given initial condition* $k_0 > 0$.

5.2.3 Converging intertemporal equilibrium

Definition 2: *An intertemporal equilibrium* $(K_t$, N_t, c_t, d_t, $m_t)_{t \geq 0}$ *is said to be converging if the sequence* $k_t = K_t/N_t$ *converges to a limit* $\overline{k} > 0$ *when* t *goes to infinity.*

Definition 3: *A converging intertemporal equilibrium* $(K_t, N_t, c_t, d_t, m_t)_{t \geq 0}$ *is said to converge in under-accumulation if* $\overline{R} > \overline{m}$. *Is is said to converge in over-accumulation if* $\overline{R} < \overline{m}$.

The example of an homothetic utility function:

$$\begin{aligned} c_t &= w_t \gamma_1(R_{t+1}) \\ \phi m_t &= w_t \gamma_3(R_{t+1}) \\ s_t &= d_t / R_{t+1} = w_t \gamma_2(R_{t+1}) \end{aligned}$$

where $\gamma_i(R)$ are such that: $\gamma_1(R) + \gamma_2(R) + \gamma_3(R) = 1$. Assumption 3 is equivalent to $\gamma_1'(R) \leq 0$ and $\gamma_3'(R) \leq 0$, which implies $\gamma_2'(R) \geq 0$.

The dynamics becomes:

$$k_{t+1} = \frac{s_t}{m_t} = \frac{\phi \gamma_2(R_{t+1})}{\gamma_3(R_{t+1})}$$

Under assumption 3, this equation has a unique solution $k_{t+1} = \overline{k}$. The economy reaches *in one period* its stationary state $\overline{k}$ defined by:

$$\overline{k} \gamma_3 \left[R(\overline{k}) \right] = \phi \gamma_2 \left[R(\overline{k}) \right]$$

- For a log linear utility

$$U_t = \gamma_1 \ln c_t + \gamma_2 \ln d_{t+1} + \gamma_3 \ln m_t$$

with $\sum_{i=1}^{3} \gamma_i = 1$,

$$\overline{k} = \frac{\phi \gamma_2}{\gamma_3}$$

- For $U_t = \left(c_t^{\frac{\rho-1}{\rho}} + \beta d_{t+1}^{\frac{\rho-1}{\rho}} + \gamma m_t^{\frac{\rho-1}{\rho}}\right)^{\frac{\rho}{\rho-1}}$ we have:

$$\gamma_2(R) = \frac{\beta^\rho R^{\rho-1}}{1 + \beta^\rho R^{\rho-1} + \gamma^\rho \phi^{1-\rho}}$$

$$\gamma_3(R) = \frac{\gamma^\rho \phi^{1-\rho}}{1 + \beta^\rho R^{\rho-1} + \gamma^\rho \phi^{1-\rho}}$$

$\overline{k}$ is such that:

$$\overline{k} R(\overline{k})^{1-\rho} = \phi^\rho \beta^\rho \gamma^{-\rho}$$

- For a CES production function,

$$F(K, L) = A\left(K^{\frac{\tau-1}{\tau}} + bL^{\frac{\tau-1}{\tau}}\right)^{\frac{\tau}{\tau-1}}$$

$\overline{k}$ exists and is unique when $\tau + \rho \geq 1$. It is the unique solution of the equation:

$$k\left(1 + bk^{-\frac{\tau-1}{\tau}}\right)^{-\frac{\rho-1}{\tau-1}} = A^{\rho-1}\phi^\rho \beta^\rho \gamma^{-\rho}$$

- For a Cobb-Douglas production function

$$F(K, L) = AK^\alpha L^{1-\alpha}$$

$\overline{k}$ is equal to:

$$\overline{k} = \left((\alpha A)^{\rho-1}\phi^\rho \beta^\rho \gamma^{-\rho}\right)^{\frac{1}{1+(1-\alpha)(\rho-1)}}$$