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Ambiguity reduction through new statistical data

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Abstract

We provide some objective foundations for a belief revision process in a situation where (i) the decision-maker's initial probabilistic knowledge is imprecise and characterized by the core of a belief function, (ii) expected new data are themselves consistent with a belief function with known focal sets and (iii) the revision process is based on belief function combination. We study the properties of the information value for such a revising in the Gilboa–Schmeidler multi-prior model. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

In this paper, we consider a decision-maker who knows that he will improve his imprecise statistical knowledge by getting some new data. This improvement will enable him to reduce the ambiguity he faces. In other words if his initial knowledge consists of a family of probability distributions, then the new data will conduct him to revise his belief and replace this family by a smaller one. This work is closely linked to [2] where a notion of information structure

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based on the idea of ambiguity being reduced is defined, and where the information value of such a process is obtained by considering the Gilboa–Schmeidler [6] Max Min Expected Utility Model. However, while in [2] the consistency of the subjective anticipations of the decision-maker about the future reduction in ambiguity was studied, no objective foundations for this process was proposed. The purpose of the present paper is to offer such an objective explanation.

The idea is the following. Initially, the decision-maker is endowed with a belief function f and considers as possible the family of probability distributions which lie in the core of f. The new statistical data he is going to receive is a belief function g compatible with f. Given this new belief function g, the decision-maker will combine his initial family $core(f)$ with the new family $core(g)$ and restrict his attention to $core(f) \cap core(g)$ since the statistical reality, i.e, the true probability distribution, necessarily belongs to the two families. The decision-maker does not know in advance which belief function he will receive. Indeed, were we to assume the contrary, he could already revise his knowledge. Yet, throughout this paper, we make the central assumption that the decision-maker knows the possible focal events of the future belief function. Thus he anticipates that he can receive any belief function g with focal events in a given set, which is compatible with f i.e. such that $core(f) \cap core(g) \neq \emptyset$. The following two examples aim at illustrating our purpose.

Example 1.1 (*Adapted from* [8]). Assume that a poll institute 1 organizes a study on how a representative sample will vote for a next election in two months. Let $S = \{a, b, c, d, e\}$ be the set of candidates. Voters' opinions, today, are not firmly established, so voters are authorized to point only a subset A of S that contains the name of the candidate they will vote for. (In order to avoid unnecessary technical complications, we assume that A must be non-empty.) Therefore institute 1 will get a belief function f, with the help of the induced objective proportions $m(A)$ for any A belonging to 2^S.

Suppose now that institute 1 learns that a few days before the actual vote, the results of a new survey performed on a similar representative sample by a poll institute 2 will be published, where the voters will be asked whether they will vote for a candidate in a given non-empty coalition $B \in 2^S$ or in the opposite non-empty coalition $\overline{B}$. (For the same reasons as above, we assume that the choice of B or $\overline{B}$ is compulsory.) Then institute 1 faces a kind of situation investigated in this paper, since it only knows the possible focal elements (here B or $\overline{B}$) of the future belief function g, and clearly the assumption of representative samples guarantees the compatibility of f and g.

In Example 1.2 we consider a more general situation in the sense that the future possible focal elements will no longer form a partition of the set S of states of nature.

Example 1.2 (*Adapted from* [7]). Let us consider a population of patients, each patient being in one of four exclusive states of health: $H =$ healthy, $D_1 =$ to suffer from disease 1, $D_2^- =$ to suffer from a mild form of disease 2, $D_2^+ =$ to suffer from a serious form of disease 2. Only medical tests permit to detect the health's states. A first medical test was applied which permits to detect with certainty D_1, i.e.: $T_1^+ = \{D_1\}$ (if the test's result is positive) and $T_1^- = \{H, D_2^-, D_2^+\}$. 30% of the population react positively. Thus $p(\{D_1\}) = 30\%$ and $p(\{H, D_2^-, D_2^+\}) = 70\%$ are the only probability we know precisely. Assume that a new medical test is now available with $T_2^+ = \{D_1, D_2^-, D_2^+\}$ (a positive results means that you are ill) and $T_2^- = \{H, D_2^-\}$ (for D_2^- the test's result can be either positive or negative). Unfortunately, because of anonymity constraint, it is not possible to combine the two test's results for each patient. Given this constraint, should we make test T_2? Yes because nevertheless we can expect a reduction of our ambiguity by combining the two test's results over the population. We will observe $x\%$ of T_2^+ and given test T_1's results, necessarily $x\% \geqslant 30\%$. From observation $x\%$, we deduce then that $p(\{D_2^-, D_2^+\}) \geqslant x - 0.3$ and $p(\{H, D_2^-\}) \geqslant 1 - x$.

Let us mention that dealing here only with belief functions (instead of, say, with convex capacities) presents several advantages. First it fits with numerous practical situations of imprecise but exact data. Second it considerably facilitates the derivation of clear-cut results essentially through the non-negativity of the Möbius inverse and the resulting simple description of the core of such capacities, a well-known result since Dempster's seminal paper [5].

The paper is organized as follows. In Section 2, we set the framework of the process we consider. In Section 3 we define the value of information. In Section 4 we study the consistency requirement for these "Ambiguity Reducing Structures". In Section 5, we compare structures in terms of informativeness.

2. Definition of the ambiguity reducing structures

We consider S a finite set of states of the world and 2^S the algebra of events of S. The decision-maker has an initial statistical knowledge $core(f)$ with f a belief function defined on 2^S and m_f its Möbius transform.

Definition 2.1. Two belief functions f and g on 2^S are said to be compatible if $core(f) \cap core(g) \neq \emptyset$.

Since the decision-maker is concerned with the real underlying probability distribution and since we suppose that the statistical data he receives is only imprecise but never wrong (i.e. the real underlying statistical situation always belongs to the family of probability distribution that sums up the knowledge of the decision-maker), we only restrict our attention to belief functions which are compatible. (For an extensive study of the combination of two compatible belief functions, see [4].)

Definition 2.2. We call $Supp(f)$ the smallest (for set inclusion) $E \subseteq S$ such that $f(E) = 1$.

For a belief function, there is no problem of existence and uniqueness of such an event.

Definition 2.3. We denote $\Phi(f) = \{E \subseteq S / m_f(E) > 0\}$ the set of focal events for the belief function f.

Obviously we have that $Supp(f) = \bigcup_{E \in \Phi(f)} E$.
Let us consider $\Sigma \subseteq 2^S$.

Definition 2.4. We denote $BF(\Sigma)$ the set of belief functions g such that $\Phi(g) \subseteq \Sigma$.

From now on we assume that the decision-maker knows that the focal events of the belief function g he will receive will all belong to Σ.

Definition 2.5. We denote $BF(\Sigma, f)$ the set of belief functions in $BF(\Sigma)$ which are compatible with f.

This definition also stands for probability distributions. In that case, we have $BF(\Sigma, p) = \{g \in BF(\Sigma) / p \in core(g)\}$.

As pointed out in Section 1, the following lemma, a central result of Dempster [5] (see also a generalization in [3] to general capacities), will be of great help in several proofs below.

Lemma 2.1. *Let f be a belief function on* $(S, 2^S)$ *with Möbius inverse* m_f *and* $\Phi(f)$ *as set of focal elements, and let p be a probability measure on* $(S, 2^S)$. *Then, the following propositions are equivalent*:

(i) $p \geqslant f$.

(ii) *There exists a mapping* $\alpha : S \times 2^S \to \mathbb{R}_+$ *such that*

(a) $\alpha(s, A) > 0 \Rightarrow s \in A$,

(b) $\sum_{s \in A} \alpha(s, A) = 1, \ \forall A \in 2^S$,

(c) $p(\{s\}) = \sum_{A \in 2^S, A \supseteq \{s\}} \alpha(s, A) \cdot m_f(A), \ \forall s \in S$.

(iii) *The same proposition as* (ii) *with* $\Phi(f)$ *instead of* 2^S.

It is also the case for the following lemma due to [4].

Lemma 2.2. *Let f and g be two belief functions, then the following two assertions are equivalent*:

(i) *The belief functions f and g are compatible.*

(ii) *There exists a mapping* $\beta : 2^S \times 2^S \to \mathbb{R}_+$ *such that*:

(a) $\beta(E,F) > 0 \Rightarrow E \times F \in \Phi(f) \times \Phi(g)$ *and* $E \cap F = \emptyset \Rightarrow \beta(E,F) = 0$,

(b) $m_f(E) = \sum_{F \in \Phi(g)} \beta(E,F)$ *and* $m_g(F) = \sum_{E \in \Phi(f)} \beta(E,F)$.

Let us add that, due to space constraints, we have only tried to emphasize the main points of the proofs of the results below, without always developing all details.

Proposition 2.1. *For all belief functions f and sets* Σ

$$BF(\Sigma, f) = \bigcup_{p \in core(f)} BF(\Sigma, p).$$

Proof. Obvious (remind that $g \in BF(\Sigma, f)$ iff $\exists p \in core(f) \cap core(g)$ and $g \in BF(\Sigma)$). $\square$

In terms of anticipations, this means that the belief functions g the decision-maker believes possible given f and Σ are the belief functions that he believes possible to receive given Σ and a possible underlying probability distribution p.

Proposition 2.2. *For all belief functions f, sets* Σ *and* $g \in BF(\Sigma, f)$, $core(f) \cap core(g) = \{p \in core(f) / g \in BF(\Sigma, p)\}$.

Proof. Obvious (this result is a mere restatement of Definitions 1 and 5). $\square$

Let us study now the decision-maker's anticipations about how he will revise his knowledge.

Definition 2.6. We denote respectively $\mathscr{F}(f)$ and $\mathscr{F}(f,g)$ the family of probability distributions in $core(f)$ and $core(f) \cap core(g)$.

Necessarily $\mathscr{F}(f,g) \subseteq \mathscr{F}(f)$ which can be interpreted as a reduction of ambiguity. [1]

Given f and Σ, a decision-maker can anticipate how his beliefs may evolve after he will get new data.

Definition 2.7. We call the set $f * \Sigma = \{\mathscr{F}(f,g) / g \in BF(\Sigma, f)\}$ an Ambiguity Reducing Structure.

[1] $\mathscr{F}(f,g)$ is what corresponds to the "revising message" considered in [2].

By definition, any $\mathscr{F}(f,g)$ in $f * \Sigma$ is not $\emptyset$.

Proposition 2.3. *Let p be given in $\mathscr{F}(f)$, then there exists at least a $\mathscr{F}(f,g) \in f * \Sigma$ such that $p \in \mathscr{F}(f,g)$ if and only if $BF(\Sigma,p) \neq \emptyset$.*

Proof. Obvious since for any $p \in \mathscr{F}(f), p \in \mathscr{F}(f,g) \Longleftrightarrow g \in BF(\Sigma,p)$. □

If there is no such $\mathscr{F}(f,g) \in f * \Sigma$, this means that the decision-maker anticipates that he will no longer consider $p$as possible whatever the data he receives.

Definition 2.8. For any $\Sigma \subseteq 2^S$ and $E \in 2^S$, we denote $\Sigma(E) = \{F \in 2^S / F = H \cap E \text{ with } H \in \Sigma\}$.

The next proposition gives the condition which ensures that the decision-maker can make some anticipations.

Proposition 2.4. *The two propositions are equivalent*
(i) $BF(\Sigma,f) \neq \emptyset$,
(ii) $\forall E \in \Phi(f),\ \Sigma(E) \neq \{\emptyset\}$.

Proof. (i) $\Rightarrow$ (ii). We show the implication by proving that if (ii) does not stand, then it is also the case for (i). Thus take an $E \in \Phi(f)$ such that $\Sigma(E) = \{\emptyset\}$. It implies that $\forall g \in BF(\Sigma),\ \forall p \in core(g), p(E) = 0$ while $\forall p \in \mathscr{F}(f),\ p(E) \geqslant m_f(E) > 0$. So $BF(\Sigma,f) = \emptyset$.

(ii) $\Rightarrow$ (i). If (ii) is true, then there exists a function $\varphi : \Phi(f) \to \Sigma$ such that $\forall E \in \Phi(f),\ E \cap \varphi(E) \neq \emptyset$. Define $g_\varphi \in BF(\Sigma)$ by $m_{g_\varphi}(F) = \sum_{E \in \Phi(f)/\varphi(E)=F} m_f(E)$. One can check that $g_\varphi \in BF(\Sigma,f)$. □

Since in this paper we assume that $BF(\Sigma,f)$ is non-empty, the above condition (ii) holds throughout the paper.

The following proposition shows an important consistency condition about the decision-maker's anticipations.

Proposition 2.5. *The two propositions are equivalent*
(i) *For all $p \in \mathscr{F}(f)$, $BF(\Sigma,p) \neq \emptyset$,*
(ii) $\bigcup_{E \in \Sigma(Supp(f))} E = Supp(f)$.

Proof. (i) $\Rightarrow$ (ii). Let us show that if (ii) does not stand, then it is also the case for (i). Suppose $\exists s \in Supp(f)$ such that $s \notin \bigcup_{E \in \Sigma(Supp(f))} E$. Then $\exists p \in \mathscr{F}(f)$ such that $p(s) > 0$. $BF(\Sigma,p) = \emptyset$ since $\forall g \in BF(\Sigma),\ s \notin Supp(g)$.

(ii) $\Rightarrow$ (i). Let us take a $p \in \mathscr{F}(f)$. Consider a function $\varphi : Supp(f) \to \Sigma$ such that $\forall s \in Supp(f),\ s \in \varphi(s)$. Define $g_\varphi \in BF(\Sigma)$ by $m_{g_\varphi}(E) = \sum_{s \in Supp(f)/\varphi(s)=E} p(s)$. We obtain $g_\varphi \in BF(\Sigma, p)$. $\square$

We can expect (ii) to be an important consistency condition. For instance, consider on the contrary that (ii) does not hold. Then there exists a p such that $BF(\Sigma, p) = \emptyset$ and, according to Proposition 2.3, this means that the decision-maker already knows that whatever g he will receive, he will not consider p as possible any more. Logically, he could eliminate ex ante p in his initial knowledge $core(f)$.

One may wonder what role Σ plays. The following results show that we can restrict our attention to $\Sigma(Supp(f))$.

Proposition 2.6. *For all belief functions f and sets $\Sigma, f * \Sigma = f * \Sigma(Supp(f))$.*

Proof. First let us prove that $f * \Sigma(Supp(f)) \subseteq f * \Sigma$. For that purpose, let us consider g belonging to $BF(\Sigma(Supp(f)), f)$, and define a new belief function $g_\varphi \in BF(\Sigma, f)$ such that $\mathscr{F}(f, g_\varphi) = \mathscr{F}(f, g)$. Thus, let φ be a mapping $E \in \Sigma(Supp(f)) \to \varphi(E) \in \Sigma$ such that $\varphi(E) \cap Supp(f) = E$, the existence of such a mapping is trivial from the definition of $\Sigma(Supp(f))$. Furthermore, it is straightforward that φ is injective. Hence let g_φ be the set-function on $(S, 2^S)$ whose Möbius inverse is defined by $m_{g_\varphi}(F) = m_g(\varphi^{-1}(F)),\ \forall F \in 2^S$. That g_φ is a belief function with support contained in Σ, can be readily checked. It remains to prove that $\mathscr{F}(f, g_\varphi) = \mathscr{F}(f, g)$.

First prove $\mathscr{F}(f, g_\varphi) \subseteq \mathscr{F}(f, g)$. Let $p \in core(f) \cap core(g_\varphi)$. That p will also belong to $core(g)$ will result through (iii) of the Lemma 2.1 from the fact that $\forall s \in Supp(p),\ \forall F \in \Sigma$, one obtains $F \supseteq \{s\}$ is equivalent to $\varphi^{-1}(F) \supseteq \{s\}$.

The converse inclusion $\mathscr{F}(f, g_\varphi) \supseteq \mathscr{F}(f, g)$, will be similarly obtained through Lemma 2.1, taking into account the fact that if $p \in \mathscr{F}(f, g)$ then $\forall s \in Supp(p),\ \forall E \in \Sigma(Supp(f))$, one obtains $E \supseteq \{s\}$ is equivalent to $\varphi(E) \supseteq \{s\}$, hence $m_{g_\varphi}(\varphi(E)) = m_g(E)$ will allow to conclude.

Conversely let us prove that $f * \Sigma \subseteq f * \Sigma(Supp(f))$. For that purpose, let us consider g belonging to $BF(\Sigma, f)$, and define a new belief function $g' \in BF(\Sigma(Supp(f)), f)$ such that $\mathscr{F}(f, g') = \mathscr{F}(f, g)$. Thus let g' be the set-function on $(S, 2^S)$ whose Möbius inverse $m_{g'}$ is defined by

$$m_{g'}(E) = \sum_{F \in \Sigma, F \cap Supp(f) = E} m_g(F), \quad \forall E \in 2^S.$$

First it is easy to show that $g \in BF(\Sigma, f)$ entails that $\forall p \in \mathscr{F}(f, g)$ and $\forall F \in \Phi(g)$, $F \cap Supp(p) \neq \emptyset$ hence $F \cap Supp(f) \neq \emptyset$, and therefore: $F \cap Supp(f) \neq \emptyset,\ \forall F \in \Phi(g)$. This will entail that $m_{g'}(\emptyset) = 0$, and that g' is actually a belief function whose support is contained in $\Sigma(Supp(f))$. It remains to prove that $\mathscr{F}(f, g) = \mathscr{F}(f, g')$.

We just confine to sketch the proof of $\mathscr{F}(f,g') \subseteq \mathscr{F}(f,g)$, the converse being similar.

Let $p \in \mathscr{F}(f,g')$. Lemma 2.1 states that there exists α' such that $\forall s \in S$: $p(\{s\}) = \sum_{G' \in 2^S, G' \supseteq \{s\}} \alpha'(s,G') \cdot m_{g'}(G')$. It follows that $\forall s \in S : p(\{s\}) = \sum_{G \in 2^S, G \supseteq \{s\}} \alpha'(s, G \cap Supp(f)) \cdot m_g(G)$. Then for a given $G \in 2^S$, defining $\alpha(s,G) = \alpha'(s, G \cap Supp(f))$ if s belongs to $G \cap Supp(f), \alpha(s,G) = 0$ if $s \in G$, $s \notin Supp(f)$, will entail through Lemma 2.1 that $p \geqslant g$, which achieves the proof. □

The decision-maker is sure that there is a zero probability that the real state of the world lies outside of $Supp(f)$, so he does not bother about these states.

Proposition 2.7. *The two propositions are equivalent*:
(i) *For all* $p \in \mathscr{F}(f)$, $BF(\Sigma(Supp(f)), p)$ *is a singleton*,
(ii) $\Sigma(Supp(f))$ *is a partition of* $Supp(f)$.

Proof. (i) $\Rightarrow$ (ii). Let us show that if (ii) does not hold, then it is also the case for (i). If (ii) does not hold, then either there exists a $s* \in Supp(f)$, a $p \in \mathscr{F}(f), E$ and F in $\Sigma(Supp(f))$ such that $p(s*) > 0, E \neq F$ and $s* \in E \cap F$, or $\bigcup_{E \in \Sigma(Supp(f))} E \subset Supp(f)$ and in this case, according to Proposition 2.5, there exists a $p \in \mathscr{F}(f)$ such that $BF(\Sigma(Supp(f)), p) = \emptyset$. Let us consider the first case. Then, there exists two functions φ_E and $\varphi_F : Supp(f) \to \Sigma(Supp(f))$ such that $\forall s \in Supp(f)$, $s \in \varphi_E(s)$ and $s \in \varphi_F(s)$ with $\varphi_E(s*) = E$, $\varphi_F(s*) = F$ and $\varphi_E(s) = \varphi_F(s)$ otherwise. Define $g_{\varphi_i} \in BF(\Sigma(Supp(f)))$ by $m_{g_{\varphi_i}}(H) = \sum_{s \in Supp(f)/\varphi_i(s) = H} p(s)$ for $i = E, F$. Then one can check that $g_{\varphi_E} \neq g_{\varphi_F}$ and $\{g_{\varphi_E}, g_{\varphi_F}\} \subseteq BF(\Sigma(Supp(f), p)$.

(ii) $\Rightarrow$ (i). Let us consider $p \in \mathscr{F}(f)$. Consider g_p defined by $m_{g_p}(E) = \sum_{s \in E} p(s)$ if $E \in \Sigma(Supp(f))$, $= 0$ otherwise. (ii) implies that g_φ is a belief function in $BF(\Sigma(Supp(f)), p)$. Take a $g \in BF(\Sigma(Supp(f), p))$. Suppose that $g \neq g_\varphi$ so that there exists a $E \in \Sigma(Supp(f))$ such that $m_g(E) \neq m_{g_\varphi}(E)$.

Suppose for instance $m_g(E) < m_{g_\varphi}(E)$. By definition of $BF(\Sigma(Supp(f)))$

$$\sum_{F \in \Sigma(Supp(f))} m_g(F) = \sum_{F \in \Sigma(Supp(f))} m_{g_\varphi}(F) = 1.$$

Then with (ii) and by definition of g_φ, this implies that $g(E) = m_g(E) < g_\varphi(E) = m_{g_\varphi}(E) = p(E)$ and

$$\begin{aligned} g(S \setminus E) &= \sum_{F \in \Sigma(Supp(f)) \setminus \{E\}} m_g(F) \\ &= 1 - m_g(E) > g_\varphi(S \setminus E) \\ &= 1 - m_{g_\varphi}(E) \\ &= p(S \setminus E) \\ &= 1 - p(E), \end{aligned}$$

which contradicts the fact that $p \in core(g)$. If $m_g(E) < m_{g_\varphi}(E)$ the contradiction comes directly from $g(E) = m_g(E) > g_\varphi(E) = m_{g_\varphi}(E) = p(E)$. $\square$

Condition (ii) is quite strong and corresponds to the particular case we have examined in the examples. The next proposition shows that there exists some weaker condition that still give some interesting property for the *Ambiguity Reducing Structures*.

Proposition 2.8. *The first two propositions are equivalent and they imply the third one*:

(i) *For all p in $\mathscr{F}(f)$, $BF(\Sigma,p) \neq \emptyset$ and for all p,p' in $\mathscr{F}(f)$, either $BF(\Sigma,p) \cap BF(\Sigma,p') = \emptyset$ or $BF(\Sigma,p) = BF(\Sigma,p')$.*

(ii) *$f * \Sigma$ is a partition of $\mathscr{F}(f)$.*

(iii) *For any $E \in \Phi(f), \Sigma(E)$ is a partition of E and for all H,G in Σ, for all $E,F \in \Phi(f)$, $H \cap E = G \cap E$ and $E \cap F \neq \emptyset$ implies $H \cap F = G \cap F$.*

Proof. (i) $\Rightarrow$ (ii). Since $\forall p$, $BF(\Sigma,p) \neq \emptyset$, then $\bigcup_{\mathscr{F}(f,g)\in f*\Sigma} \mathscr{F}(f,g) = \mathscr{F}(f)$. Consider two distinct g and g' in $BF(\Sigma,f)$ such that $\mathscr{F}(f,g) \cap \mathscr{F}(f,g') \neq \emptyset$ (however, if two such g and g' do not exist, then $f * \Sigma$ is necessarily a partition of $\mathscr{F}(f)$). Then $\exists p \in \mathscr{F}(f,g) \cap \mathscr{F}(f,g')$. $\forall p' \in \mathscr{F}(f,g), g \in BF(\Sigma,p) \cap BF(\Sigma,p')$ and (i) implies that $g' \in BF(\Sigma,p')$ and $p' \in \mathscr{F}(f,g')$. Thus $\mathscr{F}(f,g) = \mathscr{F}(f,g')$ which means that $f * \Sigma$ is a partition.

(ii) $\Rightarrow$ (i). Since $\bigcup_{\mathscr{F}(f,g)\in f*\Sigma} \mathscr{F}(f,g) = \mathscr{F}(f), \forall p \in \mathscr{F}(f)\ \exists g$ such that $p \in \mathscr{F}(f,g)$ showing that $BF(\Sigma,p) \neq \emptyset$. If $g \in BF(\Sigma,p) \cap BF(\Sigma,p')$ and $g' \in BF(\Sigma,p)$, then $p \in \mathscr{F}(f,g) \cap \mathscr{F}(f,g')$ so $\mathscr{F}(f,g) = \mathscr{F}(f,g')$ and $p' \in \mathscr{F}(f,g) \Rightarrow p' \in \mathscr{F}(f,g')$ which means that $g' \in BF(\Sigma,p')$.

(ii) $\Rightarrow$ (iii). Let us show that if (iii) does not hold, then it is also the case for (ii). If (iii) does not hold then we are in one of the following situations:

$\Sigma(E)$ is not a partition of E for all E and

(a) either $\exists E \in \Phi(F)$ such that $\bigcup_{H\in\Sigma(E)} H \subset E$ which implies

$$\bigcup_{H\in\Sigma(Supp(f))} H \subset Supp(f)$$

and by Proposition 2.5 we know then that $\bigcup_{\mathscr{F}(f,g)\in f*\Sigma} \mathscr{F}(f,g) \subset \mathscr{F}(f)$ which contradicts (ii).

(b) Or there exist two distinct F and G in $\Sigma(E)$ such that $F \cap G \neq \emptyset$. Suppose $F \setminus G \neq \emptyset$. (If $F \setminus G = \emptyset$, then necessarily $G \setminus F \neq \emptyset$ and we can adapt the proof). Then there exist functions φ_F, $\varphi_{F\cap G} : \Phi(f) \to Supp(f)$ and functions L_F, $L_G : \Phi(f) \to \Sigma$ such that $\forall H \in \Phi(f), \varphi_F(H) \in H \cap L_F(H)$, $\varphi_{F\cap G}(H) \in H \cap L_G(H)$, $\forall H \neq E$, $L_F(H) = L_G(H)$, $\varphi_F(H) = \varphi_{F\cap G}(H)$ and $\varphi_F(E) \in F \setminus G$, $\varphi_{F\cap G}(E) \in F \cap G$. Define the probability distributions p_i, for $i = F, F\cap G$ by $p_i(s) = \sum_{H\in\Phi(f)/\varphi_i(H)=s} m_f(H)$ and the belief functions g_i, for $i = F,G$ by $m_{g_i}(L) = \sum_{H\in\Phi(f)/L_i(H)=L} m_f(H)$. We can check that

$p_{F\cap G} \in \mathscr{F}(f, g_F) \cap \mathscr{F}(f, g_G)$, $p_F \in \mathscr{F}(f, g_F)$ and $p_F \notin \mathscr{F}(f, g_G)$ which means that $f * \Sigma$ is not a partition.

Or $\Sigma(E)$ is a partition of E for all E but

(c) $\exists H, G$ in Σ, E, F in $\Phi(f)$ such that $H \cap E = G \cap E$ and $H \cap F \neq G \cap F$. Then $H \cap (E \cap F) = G \cap (E \cap F) = \emptyset$ according to the partition's condition. Suppose $H \cap F \neq \emptyset$. Then, there exists $L \in \Sigma$ such that $L \cap (E \cap F) \neq \emptyset$. Then there exists a function $\psi : \Phi(f) \setminus \{E, F\} \to Supp(f)$, functions N_H, $N_G : \Phi(f) \to \Sigma$ such that $\forall M \in \Phi(f) \setminus \{E, F\}, N_H(M) = N_G(M)$ and $N_H(E) = H, N_G(E) = G, N_H(F) = N_G(F) = L$, and states $s^* \in H \cap E, s_H \in H \cap F$, $s_L \in L \cap (E \cap F)$. Define the probability distributions p_i, for $i = H, G$ by $\forall s \in S \setminus \{s^*, s_H, s_L\}, p_i(s) = \sum_{P\in\Phi(f)\setminus\{E,F\}/\psi(P)=s} m_f(P)$,

$$p_H(s^*) = \sum_{P\in\Phi(f)\setminus\{E,F\}/\psi(P)=s^*} m_f(P),$$
$$p_H(s_H) = m_f(E) + \max(0, m_f(E) - m_f(F)) + \sum_{P\in\Phi(f)\setminus\{E,F\}/\psi(P)=s^*} m_f(P),$$
$$p_H(s_L) = m_f(F) + \max(0, m_f(E) - m_f(F)) + \sum_{P\in\Phi(f)\setminus\{E,F\}/\psi(P)=s^*} m_f(P),$$
$$p_G(s^*) = m_f(E) + \sum m_f(P),$$
$$p_G(s_H) = \sum_{P\in\Phi(f)\setminus\{E,F\}/\psi(P)=s^*} m_f(P),$$
$$p_G(s_L) = m_f(F) + \sum_{P\in\Phi(f)\setminus\{E,F\}/\psi(P)=s^*} m_f(P),$$

and belief functions g_i, for $i = H, G$ by $m_{g_i}(M) = \sum_{P\in\Phi(f)/N_i(P)=M} m_f(P)$.

We can check that $p_G \in \mathscr{F}(f, g_H) \cap \mathscr{F}(f, g_G), p_H \in \mathscr{F}(f, g_H)$ and $p_H \notin \mathscr{F}(f, g_G)$ which means that $f * \Sigma$ is not a partition. $\square$

This partitional case is similar to the usual information structures conceived as partitions of [2] S. Example 1.2 was a case were condition (iii) is satisfied. This result shows that in general, $f * \Sigma$ is not a partition of $\mathscr{F}(f)$ and that the ambiguity reduction process can be itself quite fuzzy.

3. The information value of an ambiguity reducing structure

We consider the Max min EU with multi-prior model of [6] in order to analyse the value of information. Let us introduce this model of preference formally. The decision-maker is choosing between acts a which are mappings

[2] We conjecture that (iii) is also a sufficient condition to have a partition.

from S into a set of outcomes X. We suppose that the decision-maker has a utility function U defined on X and that his preference on the set of acts relies on a family $\mathscr{F}(f)$ of probability distributions on S, with f his initial belief function, through the functional

$$V_f(a) = \min_{p \in \mathscr{F}(f)} E_p U(a).$$

Without access to supplementary statistical data, the timing of decision and resolution of uncertainty is shown in Fig. 1.

Let us denote $a*$ as the optimal choice of the decision-maker when he has to choose in an opportunity set A and $V(A,f) = \max_{a \in A} V_f(a) = V_f(a*)$ the optimal value he can get.

However, after he receives a $g \in BF(\Sigma, f)$, the decision-maker chooses according to his "revised" preference $V_{f \cap g}(\cdot)$. The timing of decision and resolution of uncertainty is shown in Fig. 2.

Let us denote $a * (g)$ his optimal choice after he receives the belief function g and $V(A, f \cap g) = \max_{a \in A} V_{f \cap g}(a) = V_{f \cap g}(a * (g))$ the optimal value he gets conditionally on g. How does the decision-maker value ex ante the whole process of choosing in A after getting new statistical data? For that, we have to determine how the decision-maker evaluates ex ante the fact of getting $V(A, f \cap g)$ conditionally on g. Since the decision-maker's anticipations about the g's $\in BF(\Sigma, f)$ he may receive are totally uncertain, it seems natural to consider the following anticipated value:

$$V(A, f, \Sigma) = \min_{g \in BF(\Sigma, f)} V(A, f \cap g).$$

We will say that there is a positive value of information for the structure $f * \Sigma$ if the anticipated value $V(A, f, \Sigma)$ is greater than the value $V(A, f)$ he would get by ignoring his information possibilities.

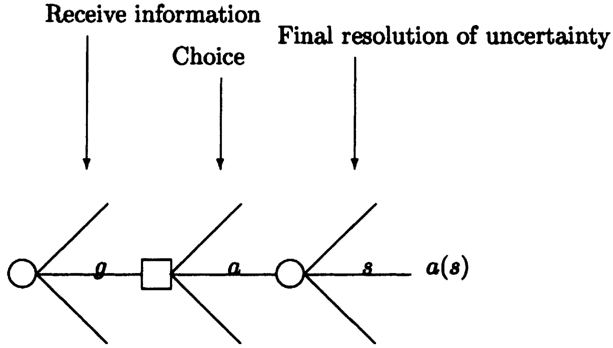

Fig. 1. No information.

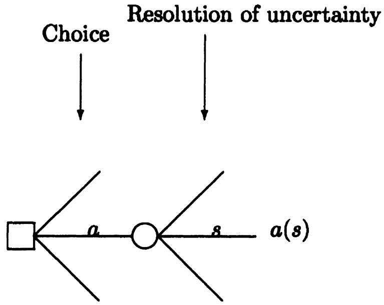

Fig. 2. With information.

Definition 3.1. The information value is equal to

$$VI(A,f,\Sigma) = V(A,f,\Sigma) - V(A,f).$$

Since $\forall E \in \Phi(f), \Sigma(E) \neq \{\emptyset\}$, hence $BF(\Sigma,f) \neq \emptyset$ and the information value is well defined.

The next results confirm the intuition that the reduction of ambiguity is positive for the decision-maker, i.e., the information value is always positive. [3]

Theorem 3.1. *For all* A, f *and* Σ, $VI(A,f,\Sigma) \geqslant 0$.

Proof. $\forall g \in BF(\Sigma,f), \mathscr{F}(f,g) \subseteq \mathscr{F}(f) \Rightarrow \forall a \in A, V(A,f \cap g) = \max_{a\in A} V_{f\cap g}(a) \geqslant V_f(a)$. So, $V(A,f\cap g) \geqslant V(A,f)$. Consequently, $V(A,f,\Sigma) = \min_{g\in BF(\Sigma,f)} V(A,f\cap g) \geqslant V(A,f)$. □

4. Consistent ambiguity reducing structure

We noted that Proposition 2.5 was about some logical consistency between the anticipation and the initial knowledge. Another way to introduce this idea of consistency between the anticipation and the initial knowledge is to consider the following notion of *Neutrality with respect to initial knowledge*.

Definition 4.1. $f * \Sigma$ satisfies Neutrality with respect to the initial knowledge $\mathscr{F}(f)$ if $\mathscr{F}(f) = \bigcup_{g\in BF(\Sigma,f)} \mathscr{F}(f,g)$.

Thus, we can complete Proposition 2.5.

[3] This positive result does not contradict the well-known result that standard "focusing" information might have a negative-value in non-Bayesian models.

Proposition 4.1. *The three propositions are equivalent*:

(i) *For all* $p \in \mathscr{F}(f),\ BF(\Sigma, p) \neq \emptyset$,

(ii) $\bigcup_{E\in\Sigma(Supp(f))} E = Supp(f)$,

(iii) $f * \Sigma$ *satisfies Neutrality with respect to the initial knowledge* $\mathscr{F}(f)$.

Proof. Proposition 2.5 stated the equivalence between (i) and (ii).

(i) $\Rightarrow$ (iii). Since $\forall p \in \mathscr{F}(f), \exists g \in BF(\Sigma, f)$ such that $p \in \mathscr{F}(f, g)$, $\mathscr{F}(f) = \bigcup_{g\in BF(\Sigma,f)} \mathscr{F}(f, g)$

(iii) $\Rightarrow$ (i). $\forall p \in \mathscr{F}(f), \exists g \in BF(\Sigma, f)$ such that $p \in \mathscr{F}(f, g)$. $\square$

Neutrality with respect to the initial knowledge captures the idea that the anticipation in an ambiguity reducing structure should not allow the decision-maker to improve ex ante his knowledge.

Let us take now a decision theoretic point of view. A positive value of information captures the decision theorists intuition that information is always valuable because it permits to adapt more accurately one's choice. Yet if the choice set is reduced to a unique act, there is no possibility of adjusting more accurately one's choice and there should be a null value of information whatever is the information structure. If on the contrary, we find a positive value of information, it is the kind of "pure" value of information that indicates that the initial knowledge does not capture all the information already available in the information structure. What is the condition that ensures that we will not find a pure value of information? The following theorem shows that the consistency conditions examined above are the right conditions.

Theorem 4.1. *The two propositions are equivalent*:

(i) $f * \Sigma$ *satisfies Neutrality with respect to the initial knowledge* $\mathscr{F}(f)$.

(ii) *For any singleton* $A, VI(A, f, \Sigma) = 0$.

Proof. (i) $\Rightarrow$ (ii). Since $A = \{a\}$

$$\begin{aligned} V(A, f, \Sigma) &= \min_{\mathscr{F}(f,g)\in f*\Sigma} \left[\min_{p\in\mathscr{F}(f,g)} E_p U(a) \right] \\ &= \min_{p\in\bigcup_{\mathscr{F}(f,g)\in f*\Sigma}\mathscr{F}(f,g)} E_p U(a) \\ &= \min_{p\in\mathscr{F}(f)} E_p U(a) \\ &= V(A, f). \end{aligned}$$

(ii) $\Rightarrow$ (i). Let us show that if (i) does not stand, then it is also the case for (ii). By Proposition 4.1, there is a $s \in Supp(f) \setminus (\bigcup_{E\in\Sigma(Supp(f))} E)$. Consider $A = \{a\}$ with a such that $U(a(s)) = \alpha$ and $\forall s* \neq s U(a(s*)) = \beta, \alpha < \beta$. (We assume the non-degeneracy of X in order to allow the construction of such an act a.) Then, since $\forall p \in \bigcup_{\mathscr{F}(f,g)\in f*\Sigma} \mathscr{F}(f,g), p(s) = 0$ it implies that $E_p U(a) = V(A,f,\Sigma) = \beta$. On the other hand, since $\exists p \in \mathscr{F}(f)$ such that $p(s) > 0, V(A,f) < \beta$ and thus $VI(A,f,\Sigma) > 0$. $\square$

5. Comparing ambiguity reducing structures

It is interesting to be able to compare ambiguity reducing structures in terms of informativeness. One way to do that, is to compare their respective values of information. Let us adapt the classical definition of [1].

Definition 5.1. $f * \Sigma$ is more informative than $f * \Sigma^*$ if for all A,

$$VI(A,f,\Sigma) \geqslant VI(A,f,\Sigma^*).$$

Our purpose is to find some equivalent comparative properties of the ambiguity reducing structures. Let us introduce the following two definitions.

Definition 5.2. $f * \Sigma$ is finer than $f * \Sigma^*$ if for all $g \in BF(\Sigma,f)$ there exists a $g^* \in BF(\Sigma^*,f)$ such that $\mathscr{F}(f,g) \subseteq \mathscr{F}(f,g^*)$.

Definition 5.3. Σ is $\Phi(f)$-finer than the set Σ^* if for all $E \in \Phi(f)$, for all $F \in \Sigma(E)$, there exists a $G \in \Sigma^*(E)$ such that $F \subseteq G$.

The following result gives the complete characterization of the partial ordering for the ambiguity reducing structures.

Theorem 5.1. *The three propositions are equivalent*:
(i) $f * \Sigma$ *is more informative than* $f * \Sigma^*$,
(ii) $f * \Sigma$ *is finer than* $f * \Sigma^*$,
(iii) Σ *is* $\Phi(f)$*-finer than the set* Σ^*.

Proof. (i) $\Rightarrow$ (iii). Let us show that if (iii) does not hold, then it is also the case for (i). Then, there exists $E \in \Phi(f)$ and $F \in \Sigma$ such that $\forall G \in \Sigma^*, E \cap F$ is not included in $E \cap G$. First,we show that there exists a $g \in BF(\Sigma,f)$ such that $\forall g^* \in BF(\Sigma^*,f), \mathscr{F}(f,g)$ is not included in $\mathscr{F}(f,g^*)$. There exists a function $\varphi : \Phi(f) \to \Sigma$ such that $\forall H \in \Phi(f), H \cap \varphi(H) \neq \emptyset, \varphi(H) = F$ if $H \cap (E \cap F) \neq \emptyset$. Define $g \in BF(\Sigma,f)$ by $\forall G \in \Sigma$ $m_g(G) = \sum_{H\in\Phi(f)/\varphi(H)=G} m_f(H)$. (Indeed, one can check that g is compatible with f.) Consider $g^* \in BF(\Sigma^*,f)$. According to Lemma 2.2, there exists a mapping $\beta^* : 2^S \times 2^S \to \mathbb{R}_+$ verifying Lemma 2.2 condition (ii). There exists a $G \in \Sigma^*$ such that $\beta^*(E,G) > 0$ and a $s \in S$ such that $s \in (E \cap F) \setminus (E \cap F \cap G)$. Then define a $p \in \mathscr{F}(f,g)$ such that

$p(\{s\}) = \sum_{H\in\Sigma/H\supseteq\{s\}} m_g(H)$. (Indeed, such a p exists.) By definition of g, we have also $p(\{s\}) = \sum_{H\in\Phi(f)/H\supseteq\{s\}} m_f(H)$. Let us check that $p \notin \mathscr{F}(f,g^*)$. If p were in $\mathscr{F}(f,g^*)$, we should have

$$\begin{aligned} p(\{s\}) &\leqslant \sum_{(H,L)\in\Phi(f)\times\Sigma^*/H\cap L\supseteq\{s\}} \beta^*(H,L) \\ &\leqslant \left(\sum_{(H,L)\in\Phi(f)\times\Sigma^*/H\supseteq\{s\}} \beta^*(H,L)\right) - \beta^*(E,G) \\ &= \left(\sum_{H\in\Phi(f)/H\supseteq\{s\}} m_f(H)\right) - \beta^*(E,G), \end{aligned}$$

which entails a contradiction.

Thus $\forall g^* \in BF(\Sigma^*,f)$, there exists a $p\in\mathscr{F}(f,g)$ with $p\notin\mathscr{F}(f,g^*)$. Let $(\alpha,\beta)\in\mathbb{R}^2$ such that $\alpha<\beta$. Since $\mathscr{F}(f,g^*)$ is a closed convex set there exists (Minkowski lemma) an hyperplan $\mathbf{H}^*$ in $\mathbb{R}^n$ going through p and with $\mathscr{F}(f,g^*)$ above $\mathbf{H}^*$. Let u be a normal to $\mathbf{H}^*$. Consider

$$\begin{aligned} u' = &\left(\frac{\beta-\alpha}{\inf_{p^*\in\mathscr{F}(f,g^*)} u\cdot p^* - u\cdot p}\right)\cdot u \\ &+ \left(\frac{\left(\alpha\cdot\inf_{p^*\in\mathscr{F}(f,g^*)} u\cdot p^*\right) - \beta\cdot(u\cdot p)}{\inf_{p^*\in\mathscr{F}(f,g^*)} u\cdot p^* - u\cdot p}\right)\cdot \mathbf{e}, \end{aligned}$$

where $\mathbf{e}$ is the unit vector of $\mathbb{R}^n$. Since we consider probability distributions, we have that $u'\cdot p = \alpha < \inf_{p^*\in\mathscr{F}(f,g^*)} u'\cdot p^* = \beta$. Let $a(g^*)$ be an act such that $U(a(g^*)) = u'$. So $u'\cdot p = E_pU(a(g^*)) = \alpha$ and $V_{f\cap g^*}(a(g^*)) = \beta$. Let us consider such an $a(g^*)$ for all $g^*\in BF(\Sigma^*,f)$ and $A = \{a(g^*)/g^*\in BF(\Sigma^*,f)\}$. Then $V(A,f,\Sigma)\leqslant V(A,f\cap g)\leqslant\alpha$ while $\forall g^*\in BF(\Sigma^*,f)$

$$V(A,f\cap g)\geqslant V_{f\cap g^*}(a(g^*)) = \beta$$

and thus $V(A,f,\Sigma^*)\geqslant\beta$. Since $VI(A,f,\Sigma) - VI(A,f,\Sigma^*) = V(A,f,\Sigma) - V(A,f,\Sigma^*)\leqslant\alpha-\beta<0$, we have exhibited a decision problem where it is better to be informed according to $f*\Sigma^*$.

(iii) $\Rightarrow$ (ii). Consider $g\in BF(\Sigma,f)$. There exists a mapping $\varphi : \Pi = \{(E,F)\in\Phi(f)\times\Sigma/E\cap F\neq\emptyset\}\to\Sigma^*$ such that $\forall(E,F)\in\Pi, E\cap F\subseteq E\cap\varphi(E,F)$. According to Lemma 2.2, there exists a mapping $\beta : 2^S\times 2^S\to\mathbb{R}_+$ verifying condition (ii) of Lemma 2.2. Define the mapping $\beta^* : 2^S\times 2^S\to\mathbb{R}_+$ by $\beta^*(E,G) = \sum_{F\in\Sigma/\varphi(E,F)=G}\beta(E,F)$ and consider g^* defined by $m_{g^*}(G) = \sum_{E\in\Phi(f)}\beta^*(E,G)$. By definition of φ and Lemma 2.2, we have $g^*\in BF(\Sigma^*,f)$. One can check using Lemma 2.1 that $\mathscr{F}(f,g)\subseteq\mathscr{F}(f,g^*)$ showing (ii).

(ii) $\Rightarrow$ (i). Let us note that $VI(A,f,\Sigma) - VI(A,f,\Sigma^*) = V(A,f,\Sigma) - V(A,f,\Sigma^*)$. Since $\forall g\in BF(\Sigma,f), \exists g^*\in BF(\Sigma^*,f)$ such that $\mathscr{F}(f,g)\subseteq\mathscr{F}(f,g^*)$, then $\forall a\in A, \min_{p\in\mathscr{F}(f,g)} E_pU(a)\geqslant\min_{p\in\mathscr{F}(f,g^*)} E_pU(a)$, so

$$V(A, f \cap g) \geqslant V(A, f \cap g^*) \geqslant V(A, f, \Sigma^*)$$

and finally $V(A, f, \Sigma) \geqslant V(A, f, \Sigma^*)$. $\square$

Note that in the particular case where Σ is a partition of S (see Example 1.1), the ambiguity reducing structure is more informative if and only if the partition is finer.

6. Concluding remarks

As emphasized in Section 1, in this paper we confine ourselves to initial imprecise probabilities situations described by belief functions, assuming moreover that ex ante the decision-maker is merely informed of the set Σ of possible focal events of a future compatible belief function. This allows us to derive in an easy way a simple characterization of such a compatible ambiguity reducing process (see Proposition 2.4). Furthermore this leads, in the framework of the multiple-priors model, both to confirm the intuition of the positiveness of ambiguity reduction (see Theorem 3.1), and to obtain a neat and meaningful characterization of the partial ordering "more informative than", in terms of fineness of "information Σ".

Assuming that ex ante information consists of a set Σ of possible focal elements may appear as a limitation, as can be shown by examining, for instance, some practical examples of opinion surveys. It will be the objective of a future paper to relax this assumption, the same will apply to the belief function hypothesis.

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