Tests for covariance stationarity and white noise, with an application to Euro/US dollar exchange rate
An approach based on the evolutionary spectral density

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Abstract

This paper proposes two non parametric tests for stationarity and white noise against the alternative of time-varying covariance structure with an application to euro/US dollar exchange rate. These tests are based on stability of evolutionary spectral density of the process. Graphical methods using the size and power, confirm the efficiency of our approach when compared with other stationarity tests, especially when data are non stationary with an approximately constant variance.

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1. Introduction

Stationarity hypothesis is often required to prove asymptotic properties of many estimators used in econometrics. Many stationarity tests exist in the literature. Pagan and Schwert (1990) proposed several non parametric tests to examine covariance stationarity in the stock market data, but their approaches examine essentially the homogeneity of the variance and this can be insufficient to detect the non stationarity of the processes which have a constant variance and a time-varying covariance (see Section 4.3, relation (4.7)). Kwiatkowski et al. (1992) proposed the well known KPSS test but it is only concerned with non stationary data with possible unit roots. This paper proposes a non parametric test for covariance stationarity and another one for the white noise, based on the stability

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of evolutionary spectral density (see, for example, Priestley and Subba Rao, 1969; Priestley, 1988; Dahlhaus, 1996; Adak, 1998). While the Sachs and Neumann’s stationarity test (2000) uses the wavelets theory framework, our approach is based on local Fourier transform. It improves the Priestley and Rao’s $\chi^2$ test (1969) and it detects other forms of non stationarity that the Pagan and Schwert’s test (1990) can do. Since the covariance function is the Fourier transform of the spectral density, our tests are sensitive to many kinds of instability on the covariance structure. Another advantage to use the evolutionary spectral density is to detect the structural change simultaneously in the frequency and time domain. In this way, Artis et al. (1992) studied the long and short run (low and high frequency) instability of the velocity of money in European countries. Section 2 gives an estimator of the evolutionary spectral density, Section 3 defines the tests, Section 4 proposes a graphical comparison with other tests and also an application to euro/US dollar exchange rate.

2. Theory of the evolutionary spectrum

2.1. Definition

The theory of the evolutionary spectrum (Priestley, 1965) is concerned with oscillatory processes, i.e., processes $\{X_t\}$ defined as follows:

$$X_t = \int_{-\pi}^{\pi} A_\omega(t) e^{i\omega t} dZ(\omega),$$

(2.1)

where, for each $\omega$, the sequence $\{A_\omega(t)\}$, as function of $t$, has a generalized Fourier transform whose modulus has an absolute maximum at the origin. $\{dZ(\omega)\}$ is an orthogonal process on $[-\pi, \pi]$ with $E[dZ(\omega)] = 0^1$, $E[|dZ(\omega)|^2] = d\mu(\omega)$ where $\mu(\omega)$ is a measure. Without loss of generality, the evolutionary spectral density of the process $\{X_t\}$ is defined by $h_\omega(\omega)$ as follows:

$$h_\omega(\omega) = \frac{dH_\omega(\omega)}{d\omega}, \quad -\pi \leq \omega \leq \pi,$$

(2.2)

where $dH_\omega(\omega) = |A_\omega(t)|^2 d\mu(\omega)$. The Priestley’s evolutionary spectrum theory is a particularly attractive concept, since it has a physical interpretation. It encompasses most other approaches as special cases and it includes many types of non stationary processes. The instantaneous variance of $\{X_t\}$ is given by:

$$\sigma^2_t = \text{var}(X_t) = \int_{-\pi}^{\pi} h_\omega(\omega) d\omega.$$

(2.3)

2.2. Estimation of the evolutionary spectrum

An estimator for $h_\omega(\omega)$ at time $t$ and frequency $\omega$, can be obtained by using two windows $\{g_u\}$ and $\{v_\omega\}$. Without loss of generality, $\hat{h}_\omega(\omega)$ is constructed as follows:

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1This condition implies that $E(X_t) = 0$.
\[ \hat{h}_j(\omega) = \sum_{\nu \in Z} w_\nu |U_{j-\nu}(\omega)|^2, \]  
(2.4)

where \( U_j(\omega) = \sum_{\nu \in Z} g_\nu X_{j-\nu} e^{-i\omega(j-\nu)} \). We choose the following windows \( \{g_\nu\} \) and \( \{w_\nu\}: \)

\[ g_\nu = \begin{cases} 1/(2\sqrt{h\pi}) & \text{if } |\nu| \leq h \\ 0 & \text{if } |\nu| > h \end{cases} \quad \text{and} \quad w_\nu = \begin{cases} 1/T' & \text{if } |\nu| \leq T'/2 \\ 0 & \text{if } |\nu| > T'/2 \end{cases}. \]  
(2.5)

Here we take \( h = 7 \) and \( T' = 20 \). From Priestley (1988) we have, \( E(\hat{h}_j(\omega)) = h_j(\omega), \ var(\hat{h}_j(\omega)) \) decreases when \( T' \) increases and:

\[ (i) |t_i - t_j| \leq T', \quad (ii) |\omega_i - \omega_j| \geq \frac{\pi}{h}. \]  
(2.6)

3. Definition of the tests

Let \( \{X_j\}_{j=1}^T \) be data from a discrete process \( \{X_j\} \) with evolutionary spectral density \( h_j(\omega) \). We consider the set of times \( \{t_j = 20i\}_{j=1}^T \), where \( I = [T/20] \) (\([.\] denotes the integer part of argument) and the set of frequencies \( \{\omega_j = \pi/20(1 + 3(j-1))\}_{j=1}^T \) which implies that \( \{t_j\} \) and \( \{\omega_j\} \) satisfy the conditions (i) and (ii) of Eq. (2.6). Let \( Y_{ij} = \log h_{ij}(\omega_j), \ h_{ij} = \log h_i(\omega_j), \ Y_i = 1/\Sigma_{j=1}^T Y_{ij}, \ Y_j = 1/\Sigma_{i=1}^T Y_{ij}, \ \mu_r = 1/\Sigma_{i=1}^T (Y_{ij} - \mu_j)^2 \) and \( S_r = 1/\hat{\sigma}^2 \Sigma_{i=1}^T (Y_{ij} - \mu_j) \) for \( r = 1, \ldots, I, \ i = 1, \ldots, T \). From Priestley (1988), we have:

\[ Y_{ij} = h_{ij} + e_{ij}, \]  
(3.1)

where the sequence \( \{e_{ij}\} \) is approximately normal, uncorrelated and identically distributed.

**Theorem 3.1.** Let \( T_i = \sup_{t=1, \ldots, |S_i|} S_i \). Then, under the null hypothesis of stationarity of \( \{X_i\} \), the limiting distribution of \( T_i \) is given by:

\[ F_i(a) = 1 - 2 \sum_{k=1}^\infty (-1)^{k+1} \exp(-2k^2 a^2). \]  
(3.2)

From Eq. (3.2), some useful critical values \( C_\alpha \), i.e. \( Pr(T_i > C_\alpha) = \alpha \), are \( C_{0.1} = 1.22, \ C_{0.05} = 1.36 \) and \( C_{0.01} = 1.63 \).

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\(^2\)This is the choice adopted by Artis et al. (1992).

\(^3\)For more details about relations (i) and (ii) of Eq. (2.6) and the choice of \( h \) and \( T' \), see Priestley and Subba Rao (1969) and Priestley (1988).
Proposition 3.2. Let

\[
T_2 = \frac{7(I - 1)I \sum_{j=1}^{7} (Y_j - \mu_j)^2}{6 \sum_{j=1}^{I} \sum_{i=1}^{I} (Y_{ij} - Y_j)^2}
\]

Suppose that \( \{X_t\} \) is stationary. Then under the null hypothesis that \( \{X_t\} \) is a white noise, \( T_2 \) has a Fisher-Snedecor distribution with 6 and \( 7(I - 1) \) degrees of freedom.

See Appendix A for the proofs of Theorem 3.1 and Proposition 3.2.

Remark 1. Theorem 3.1 investigates the null hypothesis \( H_0: h_{ij} = h_j \), i.e., the spectrum is independent of the set of times \( \{t_i\} \).

Remark 2. Proposition 3.2 tests the null hypothesis of the constancy of the spectrum, \( H_0: h_{ij} = h \).

4. Comparison with other tests by graphical methods

We use the \( P \)-value discrepancy plots and the size-power curves, defined by Davidson and MacKinnon (1993, 1994), to give some comparisons and analysis for the powers and the sizes of our tests. Following the authors, the graphs convey much more information, in a more easily assimilated form, than classical tables can do.

4.1. \( P \)-value discrepancy plots

Consider a test statistic \( \tau \) with asymptotic distribution function \( F(\tau) \). Denote by \( \{\tau_j\}_{j=1}^{N} \), \( N \) realizations of the statistic \( \tau \), generated by using a DGP\(^5\) which satisfies the null hypothesis. The \( P \) value of each \( \tau_j \) is the value \( p_j \), where:

\[
p_j = 1 - F(\tau_j) = \Pr(\tau > \tau_j). \tag{4.1}
\]

Consider now \( \hat{F}_0 \), the empirical distribution function of \( \{p_j\}_{j=1}^{N} \), defined at any point \( x_i \) in the (0,1) interval as follows:

\[
\hat{F}_0(x_i) = \frac{1}{N} \sum_{j=1}^{N} I(p_j \leq x_i), \tag{4.2}
\]

where \( I(p_j \leq x_i) \) is an indicator function taking the value 1 if the argument is true and 0 otherwise. Davidson and MacKinnon suggest the following choice of \( \{x_i\}_{i=1}^{m} \):

\[
x_i = .001, .002, \ldots, .010, .015, \ldots, .990, .991, \ldots, .999 \quad (m = 215). \tag{4.3}
\]

\(^5\)For more details, see Davidson and MacKinnon (1993, 1994).

\(^4\)Data generating process.
The **P value discrepancy plot** is the graph of $\hat{F}_0(x_i) - x_i$ against $x_i$. If the distribution of $\tau$ used to compute the $p_j$ is correct, each of the $p_j$ should be distributed as uniform (0,1). Therefore, when $\hat{F}_0(x_i) - x_i$ is plotted against $x_i$, the resulting graph should be close to the horizontal axis $y = 0$.

### 4.2. Size-power curves

The size-power curves of the statistic test $\tau$ is constructed with two empirical distribution functions $\hat{F}_0$ and $\hat{F}_1$. $\hat{F}_0$ is given by Eq. (4.2), $\hat{F}_1$ is constructed in the same way as $\hat{F}_0$ but instead of $\tau_j$ we use $N$ realizations $\{\tau_{ij}\}_{i=1}^N$ generated by using a DGP which satisfies a given alternative hypothesis. The **size-power curve** of the statistic $\tau$ is the locus of points $(\hat{F}_0(x_i), \hat{F}_1(x_i))$ when $x_i$ describes the (0,1) interval. Given two tests $\tau^{(1)}$ and $\tau^{(2)}$, if the test $\tau^{(1)}$ has a good power than the test $\tau^{(2)}$ then the size-power curve of $\tau^{(1)}$ converges more quickly to the horizontal line $y = 1$ than the one of $\tau^{(2)}$.

### 4.3. Comparison of the stationarity tests

We compare, by graphical methods, the statistic $T_1$ of Theorem 3.1 with both Pagan and Schwert’s test (PS) and the Priestley and Rao’s test (PR). Given the data $\{X_i\}_{t=1}^T$, the PS statistic is defined as follows:

$$PS = \sqrt{T} \frac{\hat{\mu}_1 - \hat{\mu}_2}{\hat{\nu}}.$$  \hspace{1cm} (4.4)

where $\hat{\mu}_1 = 2/T \Sigma_{j=1}^{T/2} X_i^j$, $\hat{\mu}_2 = 2/T \Sigma_{j=T/2+1}^{T} X_i^j$, $\hat{\nu} = \hat{\gamma}_0 + 2 \Sigma_{j=1}^{8} \hat{\gamma}_j (1 - (j/9))$ and $\{\hat{\gamma}_j\}$ is a consistent estimator of the autocovariance function of $\{X_i^2\}$. Under the null hypothesis of stationarity, the limiting distribution of $PS$ is $N(0,1)$. The PR statistic is defined as:

$$PR = \frac{21 T' \sum_{i=1}^{j} (Y_i - \mu_0)^2}{4h}, \quad (T' = 20, h = 7),$$  \hspace{1cm} (4.5)

with the same notations as Section 3. Under the null hypothesis of stationarity, the PR is distributed as $\chi^2_{(j-1)}$. The DGP under the null and the alternative hypothesis are respectively $\{y_i\}$ and $\{v_i\}$ defined as follows:

$$y_i = \rho y_{i-1} + e_i, \rho = 0.2, e_i \sim i.i.n(0,1), \quad t = 1, \ldots, 400.,$$  \hspace{1cm} (4.6)

$$v_i = c_i u_i, \quad c_i = \sqrt{1 - \rho_i^2}, \quad t = 1, \ldots, 400.,$$  \hspace{1cm} (4.7)

where $u_i = \rho_i u_{i-1} + e_i$, $e_i \sim i.i.n(0,1)$, $\rho_i = 0.7$ if $t \leq 200$ and $\rho_i = 0.4$ if $t > 200$. The evolutionary spectral densities of $\{u_i\}$ and $\{v_i\}$ are given respectively by $h^{(a)}(\omega) = 1/2 \pi |1 - \rho_i e^{-i \omega} - 2|$ and $h^{(v)}(\omega) = c_i^2 h^{(a)}(\omega)$. We can easily prove that: $var(u_i) = (1 - \rho_i^2)/(2 \pi)$ and then $var(v_i) = c_i^2 \rho_i^2$. Despite a constant variance, the process $\{v_i\}$ is non stationary because its spectrum is time dependent. The graphs are constructed from $N = 2500$ runs. Fig. 1 shows that the PS is a biased test, it has a low power against the alternative given by Eq. (4.7) and $T_1$ has a good power. Fig. 2 confirms the correct sizes of the PS and the $T_1$ statistics (their $P$-value discrepancy plots are close to
the horizontal axis, \( \sup |\hat{F}_0(x_i) - x_i| = 0.0430 \) for \( PS \) and \( \sup |\hat{F}_0(x_i) - x_i| = 0.0530 \) for \( T_1 \). In Fig. 3 we can see that the \( P \)-value discrepancy plot of the \( PR \) statistic is far from the horizontal axis \( y = 0 \) \( (\sup |\hat{F}_0(x_i) - x_i| = 0.969) \), i.e., the sizes of \( PS \) and \( T_1 \) are more satisfying than the one of \( PR \).
4.4. Size and power of the white noise test

The DGP of the null hypothesis is the white noise $\{e_t \sim i.i.n(0,1), t = 1, \ldots, 400\}$ and for the alternative hypothesis, we take the stationary process $\{y_t\}$ given by (4.6). The graphs are constructed from $N = 2500$ runs. Fig. 4 indicates a good power for $T_2$ because its size-power curve is very high and converges more quickly to the horizontal line $y = 1$. Finally Fig. 5 indicates a correct size of $T_2$ because the $P$ value discrepancy plot is close to the horizontal axis $y = 0$.

4.5. Euro/US dollar exchange rate

We apply the tests $T_1$, $PR$ and $PS$ to log-returns of euro/US dollar exchange rate $X_t = \log(p_t/p_{t-1})$ (see Fig. 6), with $\{p_t\}$ representing the daily exchange rate from 01/01/1999 to 30/04/2001. The rejection of stationarity hypothesis of $X_t$ is confirmed by $T_1 = 1.4053516$ (the critical value of $T_1$ is given by 1.36 for $\alpha = 0.05$) and $PS = -3.8372449$ (we reject the null hypothesis if $|PS| > 1.96$ for $\alpha = 0.05$) while the $PR$ statistic does not allow to reject the stationarity hypothesis. The estimation of evolutionary spectral density (Fig. 7) indicates several movements in the data (short and long run)
which can be explained by the peaks on high and low frequencies. We can also observe that the locations of the peaks are approximately stable in the frequencies axis but their amplitudes change with time. This means that $X_t$ has some characteristics of uniformly modulated process, i.e. $X_t = \sigma(t)Z_t$, where $Z_t$ is a stationary process and $\sigma(t)$ a deterministic function depending only on time.

5. Conclusion

We have used the evolutionary spectral density to construct a test for stationarity and another one for white noise. While other stationarity tests are concerned by particularly non stationary processes, our approach detects many types of instability in the covariance structure. Since the asymptotic distribution functions of our tests are obtained with $I = [T/20]$ (instead of $T$ on the usual cases), our approach is particularly useful with large samples of high-frequency data (like financial data).
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Appendix A

Proof of Theorem 3.1

Under the null hypothesis of stationarity, the spectrum is constant over time, i.e., \( h_i = h \). From the Priestley’s relation Eq. (3.1), we can write the following standard linear regression model:

\[
Y_i = h + e_i, i = 1, \ldots, I = \left[ \frac{T}{20} \right].
\]  

(A.1)

where \( \{e_i\} \) is approximately uncorrelated and identically distributed since \( \{e_{ij}\} \) is an i.i.d. sequence. The ols estimate of \( h \) is given by \( \hat{h} = 1/I \Sigma_{i=1}^{I} Y_i = 1/7 \Sigma_{i=1}^{I} \Sigma_{j=1}^{J} Y_{ij} = \mu_f \) and the ols residuals are \( \hat{e}_{ij} = Y_{ij} - \hat{h} \), thus \( S_i \) are the cumulated sums of ols residuals. Let \( B^{(I)}(z) = 1/\sigma \sqrt{I \Sigma_{i=1}^{I}} \hat{e}_{ij} \). Since the assumptions of theorem 1 of Ploberger and Kramer (1992) are obviously satisfied, the limiting distribution of \( sup_{0 \leq z \leq 1} |B^{(I)}(z)| \) is the standard Brownian bridge \( B(z) \), hence the limiting distribution of \( sup_{0 \leq z \leq 1} |B^{(I)}(z)| \) is \( sup_{0 \leq z \leq 1} |B(z)| \). From Billingsley (1968),

\[
Pr(sup_{0 \leq z \leq 1} |B(z)| > a) = 2 \sum_{k=1}^{\infty} (-1)^{k+1} \exp(-2k^2 a^2),
\]

the desired conclusion Eq. (3.2) holds since \( T = sup_{j=1, \ldots, J} |S_j| = sup_{0 \leq z \leq 1} |B^{(I)}(z)| \).

Proof of Proposition 3.2

The null hypothesis of white noise is true if the spectrum is simultaneously independent over the set of time and the set of frequency. Under the stationarity assumption, the constancy of the spectrum over time is satisfied, i.e. \( h_i = h_j \) and the model Eq. (3.1) becomes,

\[
Y_{ij} = h_i + e_{ij}, \quad i = 1, \ldots, I, \quad j = 1, \ldots, J.
\]

(A.2)

The relation (A.2) is a standard one-factor analysis of variance model. Now we can apply the classical Fisher-Snedecor’s test in model (A.2) to test the null hypothesis of the constancy of \( h_i \), i.e. \( h_j = h \), against the alternative of frequency-varying of the spectrum. The Fisher’s test is given by \( T_2 \) (see for example Davidson and Mackinnon, 1993).
References


